# THE LAWS OF COSINES FOR NON-EUCLIDEAN TETRAHEDRA

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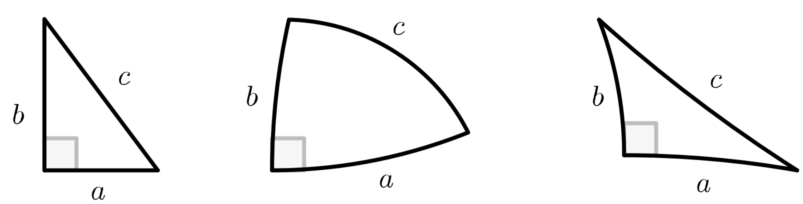
Darko Veljan's article[1] "The 2500-Year-Old Pythagorean Theorem" discusses the history and lore of "probably the only nontrivial theorem in mathematics that most people know by heart".




$a^2 + b^2 = c^2$ $\qquad$ $\cos a \cos b = \cos c$ $\qquad$ $\cosh a \cosh b = \cosh c$

Figure 1. The Pythagorean Theorems of Euclidean, spherical, and hyperbolic geometry

Veljan mentions many relatives of "the Theorem" in various contexts. Of interest here are the counterparts in non-Euclidean geometry (Figure 1), the Law of Cosines generalizations to arbitrary triangles (Figure 2), and de Gua's analogue for "right-corner" Euclidean tetrahedra (Figure 3). Tetrahedral Laws of Cosines (also Figure 3), while omitted from Veljan's article, are known, are readily proven (they're simply vector identities), and have straightforward analogues in all higher dimensions. The purpose of this note is to present novel(?)[2] analogous, generalized counterpart results for right-corner and arbitrary non-Euclidean tetrahedra.

## 1. Preliminaries

1.1. **Notation.** To streamline and unify our non-Euclidean formulas, we employ a notational "Morse Code" —dots for cosines and dashes for sines— defined thusly:

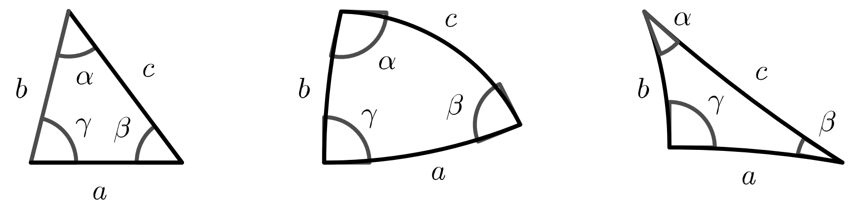
$$\ddot{x} := \cos x \ \text{ or } \ \cosh x \qquad\qquad \overline{x} := \sin x \ \text{ or } \ \sinh x$$

The context of $x$ dictates the interpretation. (For $x$ an angle, a spherical length. or a non-Euclidean area, $\ddot{x}$ and $\overline{x}$ are circular functions; for $x$ a hyperbolic length, they're hyperbolic functions.) To

*Date*: 1 January, 2005; updated 30 January, 2006, to include the Second Law of Cosines. Revised, 30 May, 2022, in part to rename the First and Second Laws as the "Law of Concurrent Cosines" and the "Law of Opposite Cosines".

[1] *Mathematics Magazine*, Vol. 73, No. 4. (October, 2000.)

[2] This author has posted these results a time or two to online discussion groups; they seem to have been unknown to the mathematical community before then.

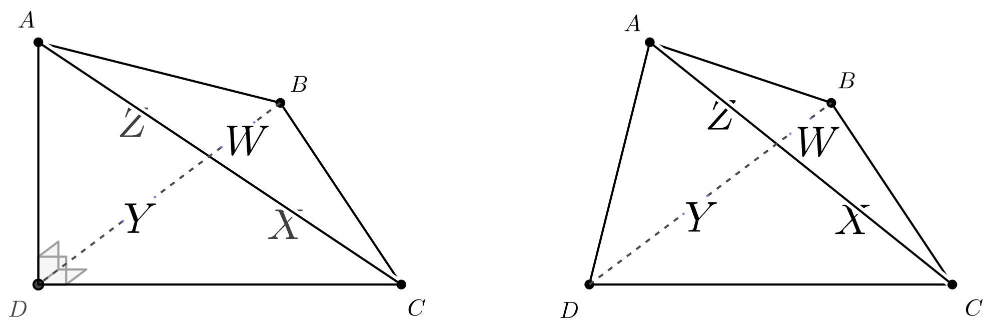



$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \text{(Euclidean)}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos\gamma, \quad \cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c \quad \text{(spherical)}$$

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma, \quad \cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cosh c \quad \text{(hyperbolic)}$$

FIGURE 2. Assorted Laws of Cosines for triangles

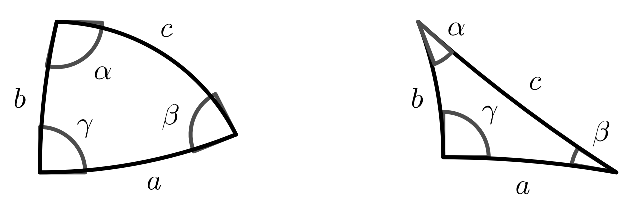



$$W^2 = X^2 + Y^2 + Z^2$$

$$W^2 = X^2 + Y^2 + Z^2 - 2YZ\cos\angle DA - 2ZX\cos\angle DB - 2XY\cos\angle DC$$

$$Y^2 + Z^2 - 2YZ\cos\angle DA = W^2 + X^2 - 2WX\cos\angle BC$$

FIGURE 3. The Pythagorean (de Gua) Theorem and Laws of Cosines for Euclidean tetrahedra




$$\text{area} = (\alpha + \beta + \gamma) - \pi \qquad\qquad \text{area} = \pi - (\alpha + \beta + \gamma)$$

FIGURE 4. Formulas for the area of a non-Euclidean triangle

account for the occasional sign change, we agree that, in "±" and "∓", the top sign applies to the spherical case, and the bottom to the hyperbolic case. With these conventions spherical and hyperbolic instances of relations can be expressed compactly and simultaneously; for instance,[3]

$$\begin{aligned} \text{area} &= \pm(\alpha+\beta+\gamma) \mp \pi \\ \ddot{c} &= \ddot{a}\ddot{b} \pm \overline{a}\overline{b}\ddot{\gamma} && \text{(Law of Cosines for Sides)} \\ \ddot{\gamma} &= -\ddot{\alpha}\ddot{\beta} + \overline{\alpha}\overline{\beta}\ddot{c} && \text{(Law of Cosines for Angles)} \end{aligned}$$

Also, to reduce visual clutter when referencing half-angles, -sides, and -areas, we write $x_2$ for $x/2$.

1.2. **A Heron-like Area Formula.** A standard formula for the area of a non-Euclidean triangle in terms of its side-lengths is this:

$$\cos\tfrac{1}{2}(\text{area}) = \frac{1+\ddot{a}+\ddot{b}+\ddot{c}}{4\ddot{a}_2\ddot{b}_2\ddot{c}_2} \tag{1}$$

It follows —as the reader may verify— from applying the Laws of Cosines and trigonometric half-argument identities to the expanded form of the angle-based relation:

$$\cos\tfrac{1}{2}(\text{area}) = \cos\tfrac{1}{2}\left(\pm(\alpha+\beta+\gamma)\mp\pi\right) = \sin(\alpha_2+\beta_2+\gamma_2)$$

1.3. **The Heron-like Formula in Tetrahedral Context.** Let $\triangle ABC$ be the face of a tetrahedron with fourth vertex $D$ (as in Figure 3). We can express the triangle's area (call it $W$) in terms of the edges and face-angles surrounding $D$. Define the following:

$$\begin{aligned} x &:= |DA| & y &:= |DB| & z &:= |DC| \\ \theta &:= \angle BDC & \phi &:= \angle CDA & \psi &:= \angle ADB \end{aligned} \tag{2}$$

We simply substitute from the Law of Cosines for Sides ($\ddot{a} = \ddot{y}\ddot{z} \pm \overline{y}\,\overline{z}\,\ddot{\theta}$, etc) into (1):

$$\ddot{W}_2 := \cos\tfrac{1}{2}W = \frac{1+\ddot{y}\ddot{z}+\ddot{z}\ddot{x}+\ddot{x}\ddot{y} \pm \left(\overline{y}\,\overline{z}\,\ddot{\theta} + \overline{z}\,\overline{x}\,\ddot{\phi} + \overline{x}\,\overline{y}\,\ddot{\psi}\right)}{4\ddot{a}_2\,\ddot{b}_2\,\ddot{c}_2} \tag{3}$$

This note's featured "hedronometric" Laws of Cosines arise from eliminating the edge and face-angle terms in favor of face-area and dihedral angle terms.

## 2. The Tetrahedral Laws of Cosines

2.1. **The "Law of Concurrent Cosines".** The Law of Cosines for dihedral angles along concurrent edges of a non-Euclidean tetrahedron is as follows (where, for simplicity, we write $A$, $B$, $C$ for the dihedral angles along edges $DA$, $DB$, $DC$):

$$\begin{aligned} \ddot{W}_2 &= \ddot{X}_2\ddot{Y}_2\ddot{Z}_2 \pm \overline{X_2Y_2Z_2}S + \ddot{X}_2\overline{Y_2}\,\overline{Z_2}\ddot{A} + \overline{X_2}\ddot{Y}_2\overline{Z_2}\ddot{B} + \overline{X_2Y_2}\ddot{Z}_2\ddot{C} \\ S &:= \sqrt{1-2\ddot{A}\ddot{B}\ddot{C}-\ddot{A}^2-\ddot{B}^2-\ddot{C}^2} \end{aligned} \tag{LoCC}$$

This result reduces to a de Gua-like counterpart for right-corner tetrahedra:

[3]The first states the convenient fact that non-Euclidean triangle's area is a simple function of its angle sum. For spherical geometry, the area is the *angular excess*: the amount by which the sum exceeds $\pi$. For hyperbolic geometry, the area is the *angular defect*: the amount by which the sum falls short of $\pi$.

**Corollary.** *If $A = B = C = \pi/2$, then*

$$\ddot{W}_2 = \ddot{X}_2\ddot{Y}_2\ddot{Z}_2 \pm \overline{X_2Y_2Z_2}$$

To prove (LoCC), we take $x$, $y$, $z$, $\theta$, $\phi$, $\psi$ as defined in (2) and expanding the half-area elements on the right-hand side in terms of sides and face-angles surrounding vertex $D$:

$$\ddot{X}_2 = \frac{1+\ddot{y}+\ddot{z}+\ddot{a}}{4\ddot{y}_2\ddot{z}_2\ddot{a}_2} = \frac{1+\ddot{y}+\ddot{z}+\ddot{y}\ddot{z}\pm\overline{yz}\ddot{\theta}}{4\ddot{y}_2\ddot{z}_2\ddot{a}_2} \qquad \overline{X_2} = \frac{\overline{y}\,\overline{z}\,\overline{\theta}}{4\ddot{y}_2\ddot{z}_2\ddot{a}_2}$$

$$\ddot{Y}_2 = \frac{1+\ddot{z}+\ddot{x}+\ddot{z}\ddot{x}\pm\overline{zx}\ddot{\phi}}{4\ddot{z}_2\ddot{x}_2\ddot{b}_2} \qquad \overline{Y_2} = \frac{\overline{z}\,\overline{x}\,\overline{\phi}}{4\ddot{z}_2\ddot{x}_2\ddot{b}_2}$$

$$\ddot{Z}_2 = \frac{1+\ddot{x}+\ddot{y}+\ddot{x}\ddot{y}\pm\overline{xy}\ddot{\psi}}{4\ddot{x}_2\ddot{y}_2\ddot{c}_2} \qquad \overline{Z_2} = \frac{\overline{x}\,\overline{y}\,\overline{\psi}}{4\ddot{x}_2\ddot{y}_2\ddot{c}_2}$$

Relations among face- and dihedral angles about a vertex match across all flavors of geometry.[4]

$$\ddot{A} = \frac{\ddot{\theta}-\ddot{\phi}\ddot{\psi}}{\overline{\phi\psi}} \qquad \ddot{B} = \frac{\ddot{\phi}-\ddot{\psi}\ddot{\theta}}{\overline{\psi\theta}} \qquad \ddot{C} = \frac{\ddot{\psi}-\ddot{\theta}\ddot{\phi}}{\overline{\theta\phi}} \qquad S = \frac{1+2\ddot{\theta}\ddot{\phi}\ddot{\psi}-\ddot{\theta}^2-\ddot{\phi}^2-\ddot{\psi}^2}{\overline{\theta\phi\psi}}$$

With these, reducing the right-hand side of (LoCC) to the left-hand side (in the form of (3)) is merely a matter of tedious algebraic manipulation and simplification, subject (fittingly) to the Pythagorean trigonometric identities

$$\ddot{x}^2 \pm \overline{x}^2 = \cos^2 x + \sin^2 x = \cosh^2 x - \sinh^2 x \equiv 1$$

2.2. **The "Law of Opposite Cosines".** The Law of Cosines for dihedral angles along opposite edges is as follows (writing $A'$, $B'$, $C'$ for the dihedral angles along edges $BC$, $CA$, $AB$):

$$\text{(LoOC)} \qquad \ddot{Y}_2\ddot{Z}_2 + \overline{Y_2Z_2}\ddot{A} \;=\; \ddot{W}_2\ddot{X}_2 + \overline{W_2X_2}\ddot{A}'$$

Before verifying the relation, we observe an immediate consequence.[5]

**Corollary.** *In an ideal hyperbolic tetrahedron —that is, one with each of its vertices at infinity— dihedral angles along opposite edges are congruent.*

Now, we demonstrate (LoOC) by revealing the symmetric nature of the relation's left-hand side. Formulas from preceding sections give us the following (after some simplification):

$$\ddot{Y}_2\ddot{Z}_2 + \overline{Y_2Z_2}\ddot{A} = \frac{(1+\ddot{x}+\ddot{z}+\ddot{b})(1+\ddot{x}+\ddot{y}+\ddot{c}) + \overline{x}^2\overline{yz}(\ddot{\theta}-\ddot{\phi}\ddot{\psi})}{16\ddot{x}_2{}^2\ddot{y}_2\ddot{z}_2\ddot{b}_2\ddot{c}_2}$$

Using the (triangular) Law of Cosines in appropriate faces, we re-write the numerator's final term without face-angles or sines:

$$\overline{x}^2\overline{yz}(\ddot{\theta}-\ddot{\phi}\ddot{\psi}) \;=\; \overline{x}^2(\overline{yz}\ddot{\theta}) - (\overline{xz}\ddot{\phi})(\overline{xz}\ddot{\psi}) \;=\; (1-\ddot{x}^2)(\ddot{a}-\ddot{y}\ddot{z}) - (\ddot{b}-\ddot{x}\ddot{z})(\ddot{c}-\ddot{x}\ddot{y})$$

[4]Spherical and hyperbolic geometry are both "locally Euclidean", and angle measure is a local notion.

[5]This follows from the fact that an ideal hyperbolic *triangle* has all angle measures 0, hence area $\pi$. Therefore, the LoOC for an ideal tetrahedron equates cosines of opposite dihedral angles.

Consequently, we have the following form:

$$\ddot{Y}_2\ddot{Z}_2 + \overline{Y_2Z_2}\ddot{A} = \frac{1 + \ddot{a} + \ddot{b} + \ddot{c} + \ddot{x} + \ddot{y} + \ddot{z} - \ddot{a}\ddot{x} + \ddot{b}\ddot{y} + \ddot{c}\ddot{z}}{8\ddot{y}_2\ddot{z}_2\ddot{b}_2\ddot{c}_2} \tag{4}$$

The right-hand side is symmetric with respect to switching opposite edges $a \leftrightarrow x$ and $b \leftrightarrow y$, which implies that the left-hand side is symmetric with respect to switching certain faces and angles:

$$\underset{\triangle xbz}{Y} \leftrightarrow \underset{\triangle ayz}{X} \qquad \underset{\triangle xyc}{Z} \leftrightarrow \underset{\triangle abc}{W} \qquad \underset{\text{edge } x}{A} \leftrightarrow \underset{\text{edge } a}{A'}$$

This proves the result.

### 2.3. **Pseudo-Faces and the "Law of Face Cosines".**

As in the Euclidean case discussed elsewhere by this author, the Law of Opposite Cosines invites the formal definition of "pseudo-faces" with areas $H$, $J$, $K$ satisfying relations that make the LoOC look more like a triangular Law.[6]

$$\begin{aligned}
\ddot{Y}_2\ddot{Z}_2 + \overline{Y_2Z_2}\ddot{A} &= \ddot{H}_2 = \ddot{W}_2\ddot{X}_2 + \overline{W_2X_2}\ddot{A}' \\
\ddot{Z}_2\ddot{X}_2 + \overline{Z_2X_2}\ddot{B} &= \ddot{J}_2 = \ddot{W}_2\ddot{Y}_2 + \overline{W_2Y_2}\ddot{B}' \\
\ddot{X}_2\ddot{Y}_2 + \overline{X_2Y_2}\ddot{C} &= \ddot{K}_2 = \ddot{W}_2\ddot{Z}_2 + \overline{W_2Z_2}\ddot{C}'
\end{aligned}$$

Unlike with the Euclidean case, this author has not yet determined if such pseudo-faces have a geometric interpretation. (Whether they might provide any insights with respect to a Heron-like formula for volume is not at all clear.)

Pseudo-faces help re-write Law of Concurrent Cosines in a symmetric, face-agnostic form that, as a bonus, avoids reference to the quantity $S$. We dub this counterpart of the Euclidean Sum-of-Squares identity ($W^2 + X^2 + Y^2 + Z^2 = H^2 + J^2 + K^2$) the Law of Face Cosines:

$$\begin{aligned}
0 = 1 &- \ddot{W}_2^{\,2} - \ddot{X}_2^{\,2} - \ddot{Y}_2^{\,2} - \ddot{Z}_2^{\,2} - 4\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2 \\
&- \ddot{H}_2^{\,2} - \ddot{J}_2^{\,2} - \ddot{K}_2^{\,2} - 2\ddot{H}_2\ddot{J}_2\ddot{K}_2 \\
&+ 2\ddot{H}_2(\ddot{W}_2\ddot{X}_2 + \ddot{Y}_2\ddot{Z}_2) + 2\ddot{J}_2(\ddot{W}_2\ddot{Y}_2 + \ddot{Z}_2\ddot{X}_2) + 2\ddot{K}_2(\ddot{W}_2\ddot{Z}_2 + \ddot{X}_2\ddot{Y}_2)
\end{aligned} \tag{LoFC}$$

We arrive at this identity by isolating $S$ in (LoCC), squaring, and using the defining relations for $H$, $J$, $K$ to rewrite dihedral angle cosines in terms of face and pseudo-face areas.

The instances of the LoCC and LoOC can be recovered by making appropriate substitutions for the $H$, $J$, $K$ terms and solving the resulting quadratic. Indeed, many additional relations —for a total of $26,244$— arise from mixing and matching those substitutions.[7]

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[6] Note that (4) gives a way to compute a pseudo-face area from the tetrahedron's edges.

[7] Arithmetic check. There are three different ways to substitute into $\ddot{H}_2^{\,2}$: using an expression in $A$ for both factors, or $A'$ for both factors, or one each of $A$ and $A'$. Likewise, there are 3 ways each to substitute into $\ddot{J}_2^{\,2}$, $\ddot{K}_2^{\,2}$, $2\ddot{H}_2$, $2\ddot{J}_2$, $2\ddot{K}_2$, and 36 ways to substitute into $2\ddot{H}_2\ddot{J}_2\ddot{K}_2$. Thus, the total number of relations is $3^6 \cdot 36 = 26,244$.

## 3. Remark

Infinitesimally, the Laws of Cosines for non-Euclidean tetrahedra become the Laws of Cosines for Euclidean tetrahedra. We demonstrate this by expressing each half-area element with a power series, and ignoring terms above the minimal degree.

For the Law of Concurrent Cosines, we have

$$\begin{aligned} 1-\tfrac{1}{2}W_2^2 \;&\sim\; \left(1-\tfrac{1}{2}X_2^2\right)\left(1-\tfrac{1}{2}Y_2^2\right)\left(1-\tfrac{1}{2}Z_2^2\right) \pm (X_2)\,(Y_2)\,(Z_2)\,S + \left(1-\tfrac{1}{2}X_2^2\right)(Y_2)\,(Z_2)\,\ddot{A} + \cdots \\ &\sim\; \left(1-\tfrac{1}{8}X^2-\tfrac{1}{8}Y^2-\tfrac{1}{8}Z^2\right) \pm (0)S + \tfrac{1}{4}YZ\ddot{A} + \tfrac{1}{4}ZX\ddot{B} + \tfrac{1}{4}XY\ddot{C} \end{aligned}$$

whence

$$W^2 \;\sim\; X^2+Y^2+Z^2-2YZ\cos A-2ZX\cos B-2XY\cos C$$

Likewise for the Law of Opposite Cosines,

$$\begin{aligned} \left(1-\tfrac{1}{2}Y_2^2\right)\left(1-\tfrac{1}{2}Z_2^2\right)+(Y_2)\,(Z_2)\,\ddot{A} \;&\sim\; \left(1-\tfrac{1}{2}W_2^2\right)\left(1-\tfrac{1}{2}X_2^2\right)+(W_2)\,(X_2)\,\ddot{A}' \\ \left(1-\tfrac{1}{8}Y^2-\tfrac{1}{8}Z^2\right)+\tfrac{1}{4}YZ\ddot{A} \;&\sim\; \left(1-\tfrac{1}{8}W^2-\tfrac{1}{8}X^2\right)+\tfrac{1}{4}WX\ddot{A}' \\ Y^2+Z^2-2YZ\cos A \;&\sim\; W^2+X^2-2WX\cos A' \end{aligned}$$

The reader can verify that the Law of Face Cosines reduces to

$$\left(W^2+X^2+Y^2+Z^2-H^2-J^2-K^2\right)^2 \;\sim\; 0$$