PSEUDOFACES OF TETRAHEDRA

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The Law of Cosines for tetrahedron $OABC$ states that

$$W^2 = X^2 + Y^2 + Z^2 - 2YZ\cos\angle OA - 2ZX\cos\angle OB - 2XY\cos\angle OC \tag{1}$$

where

$$W := |\triangle ABC| \qquad X := |\triangle OBC| \qquad Y := |\triangle OCA| \qquad Z := |\triangle OAB|$$

and where $\angle OA$, $\angle OB$, $\angle OC$ are the dihedral angles along respective edges. (To reduce visual clutter going forward, we'll denote these angles as simply A, B, C. Similarly, we'll denote dihedral angles along edges BC, CA, AB as D, E, F.) The resemblance to the Law of Cosines for a triangle is already striking, but can be made moreso.

Combining (1) with its counterparts

$$\begin{aligned}
X^2 &= W^2 + Y^2 + Z^2 - 2\,YZ\cos A - 2ZW\cos F - 2WY\cos E \\
Y^2 &= X^2 + W^2 + Z^2 - 2WZ\cos F - 2\,ZX\cos B - 2XW\cos D \\
Z^2 &= X^2 + Y^2 + W^2 - 2YW\cos E - 2WX\cos D - 2XY\cos C
\end{aligned}$$

yields a family of relations

$$\begin{aligned}
Y^2 + Z^2 - 2YZ\cos A &= W^2 + X^2 - 2WX\cos D \\
Z^2 + X^2 - 2ZX\cos B &= W^2 + Y^2 - 2WY\cos E \\
X^2 + Y^2 - 2XY\cos C &= W^2 + Z^2 - 2WZ\cos F
\end{aligned} \tag{2}$$

that teeter on the brink of exactly matching the triangular Law. We push them over that brink by formally defining H, J, K such that

$$\begin{aligned}
Y^2 + Z^2 - 2YZ\cos A \;&=\; H^2 \;=\; W^2 + X^2 - 2WX\cos D \\
Z^2 + X^2 - 2ZX\cos B \;&=\; J^2 \;=\; W^2 + Y^2 - 2WY\cos E \\
X^2 + Y^2 - 2XY\cos C \;&=\; K^2 \;=\; W^2 + Z^2 - 2WZ\cos F
\end{aligned} \tag{3}$$

We say that H, J, K are the areas of the tetrahedron's "pseudofaces" ... whatever *that* might mean. Note that, although *pseudo*, these faces are collectively as good as *real* faces (at least when squared):

$$W^2 + X^2 + Y^2 + Z^2 = H^2 + J^2 + K^2 \tag{4}$$

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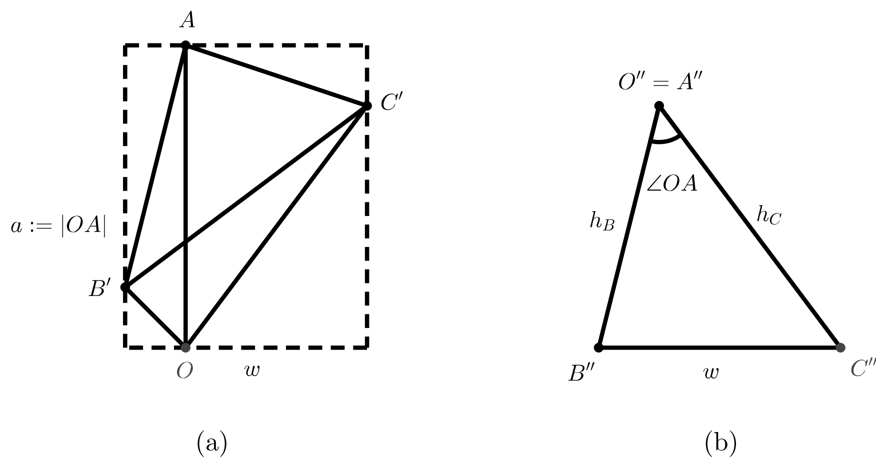

FIGURE 1. Views of a Tetrahedron. (a) Perpendicular to OA and BC; (b) parallel to OA

With judicious substitution and rearrangement, one can use pseudofaces to convert this "hedronometric" volume formula that favors one vertex (here, O):

$$V^4 = \frac{4}{81} X^2 Y^2 Z^2 \left(1 - 2\cos A \cos B \cos C - \cos^2 A - \cos^2 B - \cos^2 C\right) \tag{5}$$

into something more symmetrical

$$\begin{aligned} 81V^4 &= 2W^2X^2Y^2 + 2W^2X^2Z^2 + 2W^2Y^2Z^2 + 2X^2Y^2Z^2 + H^2J^2K^2 \\ &\quad - H^2(W^2X^2 + Y^2Z^2) - J^2(W^2Y^2 + Z^2X^2) - K^2(W^2Z^2 + X^2Y^2) \end{aligned} \tag{6}$$

So, here we see that the collection of faces and pseudofaces determine a tetrahedron (and its volume) uniquely. This type of result isn't true for just faces alone.

Pseudofaces Revealed

Although initially defined merely to make a few equations look nice, pseudofaces in fact have a geometric interpretation: viewing a tetrahedron in a direction perpendicular to the plane determined by opposite edges (for instance OA and BC, as in Figure (1a) below) reveals a quadrilateral. That quadrilateral is a pseudoface, insofar as its area is computed via a pseudoface formula.

Figure (1a) surrounds a pseudoface with a rectangle of dimensions $a := |OA|$ and some w, so that the area satisfies

$$H^2 = \frac{1}{4} a^2 w^2$$

From Figure (1b), we compute length w from the triangular Law of Cosines, using the dihedral angle $\angle OA$ (again, represented simply as A) and lengths h_B and h_C (which are the altitudes of faces $\triangle OAB$ and $\triangle OAC$, respectively):

$$w^2 = h_B^2 + h_C^2 - 2h_B\, h_C\, \cos A$$

Thus,

$$\begin{aligned} H^2 &= \frac{1}{4}a^2\left(h_B^2 + h_C^2 - 2h_B\,h_C\,\cos A\right) \\ &= \left(\frac{1}{2}ah_B\right)^2 + \left(\frac{1}{2}ah_C\right)^2 - 2\left(\frac{1}{2}ah_B\right)\left(\frac{1}{2}ah_C\right)\cos A \\ &= Y^2 + Z^2 - 2YZ\cos A \end{aligned}$$

A slightly different analysis, involving sighting the tetrahedron along BC, would yield

$$H^2 = W^2 + X^2 - 2WX\cos D$$

The geometrically-defined H, then, matches the expediently-defined H in (3). Projecting the tetrahedron into planes determined by the remaining two pairs of opposite edges give the counterpart relations for J and K.

Assorted Pseudoface Formulas

A Volume Formula. If h is the distance between edges OA and BC —the "pseudo-altitude" corresponding to pseudoface H— then

$$3V = hH \tag{7}$$

Proof. Project points B and C to B' and C' in the plane through OA parallel to BC, and project points O and A to O' and A' in the plane through BC parallel to OA. Then $OB'AC'$ and $O'BA'C$ are copies of pseudoface H, and form bases of a right prism with height h and volume hH. We can carve tetrahedron $OABC$ out of the prism by removing the height-p tetrahedron pairs $OO'BC$ and $AA'BC$ (whose bases together form $O'BA'C$) and $BB'OA$ and $CC'OA$ (whose bases together form $O'BA'C$). Each of these tetrahedron pairs has one-third the volume of the prism, so that $OABC$ makes up the last third of that volume. ☐

Cosine Formulas. We naturally define "the" angle between two (real) faces of a tetrahedron as the dihedral angle containing the interior of the tetrahedron (or as the supplement of the angle between two inward-pointing —or two outward-pointing— normal vectors). Pseudofaces have no inherent position relative to the interior of the tetrahedron, and so we cannot define the angle between pseudofaces without ambiguity or artifice. We will accept the ambiguity and let vectors provide the artifice.

Recall that, for vectors $\mathbf{u}$ and $\mathbf{v}$ subtending angle θ,

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \qquad \text{and} \qquad \|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta \tag{8}$$

and that $\mathbf{u}\times\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Thus, we have formulas such as these:

$$X^2 = \frac{1}{4}\|\mathbf{b}\times\mathbf{c}\|^2, \quad Y^2 = \frac{1}{4}\|\mathbf{c}\times\mathbf{a}\|^2, \quad Z^2 = \frac{1}{4}\|\mathbf{a}\times\mathbf{b}\|^2, \quad W^2 = \frac{1}{4}\|(\mathbf{a}-\mathbf{b})\times(\mathbf{c}-\mathbf{b})\|^2$$

$$YZ\cos A = -\frac{1}{4}(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{a}\times\mathbf{b}), \qquad ZX\cos B = -\frac{1}{4}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c}),$$

$$XY\cos C = -\frac{1}{4}(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}\times\mathbf{a})$$

where $\mathbf{a} := \overrightarrow{OA}$, $\mathbf{b} := \overrightarrow{OB}$, $\mathbf{c} := \overrightarrow{OC}$.

(Note that, although there are more direct approaches, one can recover the Law of Cosines for Tetrahedra by expanding the vector formula for W^2 in terms of the other expressions.)

With pseudofaces, we have these unambiguous area formulas:

$$H^2 = \frac{1}{4}\|\mathbf{a} \times (\mathbf{c} - \mathbf{b})\|^2 \qquad J^2 = \frac{1}{4}\|\mathbf{b} \times (\mathbf{a} - \mathbf{c})\|^2 \qquad K^2 = \frac{1}{4}\|\mathbf{c} \times (\mathbf{b} - \mathbf{a})\|^2 \tag{9}$$

For angles, we make a subjective selection of normal vector (which depends upon our preference for vertex O), and declare the angle between two pseudofaces is the angle (not the supplement of the angle) between the two corresponding normal vectors. In particular, writing θ_H for the preferred angle between pseudofaces J and K, we have

$$\begin{aligned} JK \cos\theta_H &:= -\frac{1}{4}\,(\,\mathbf{b} \times (\mathbf{a} - \mathbf{c})\,) \cdot (\,\mathbf{c} \times (\mathbf{b} - \mathbf{a})\,) \\ &= \cdots \\ &= -\frac{1}{4}\begin{pmatrix} (\mathbf{c} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{a}) \cdot (\mathbf{a} \times \mathbf{c}) \\ +(\mathbf{c} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{c} \times \mathbf{b}) \end{pmatrix} \\ &= -X^2 + YZ\cos A + ZX\cos B + XY\cos C \end{aligned}$$

so that

$$\begin{aligned} 2JK\cos\theta_H &= -2X^2 + (2YZ\cos A + 2ZX\cos B + 2XY\cos C) \\ &= -2X^2 + (X^2 + Y^2 + Z^2 - W^2) \\ &= (Y^2 + Z^2) - (W^2 + X^2) \\ &= (H^2 + 2YZ\cos A) - (H^2 + 2WX\cos D) \\ &= 2(YZ\cos A - WX\cos D) \end{aligned}$$

Altogether, we have

$$\begin{aligned} (Y^2 + Z^2) - (W^2 + X^2) \;&=\; 2JK\cos\theta_H \;=\; 2(YZ\cos A - WX\cos D) \\ (Z^2 + X^2) - (W^2 + Y^2) \;&=\; 2KH\cos\theta_J \;=\; 2(ZX\cos B - WY\cos E) \\ (X^2 + Y^2) - (W^2 + X^2) \;&=\; 2HJ\cos\theta_K \;=\; 2(XY\cos C - WZ\cos F) \end{aligned}$$

With other choices in how we define the angles θ_H, θ_J, θ_K, the above formulas could be off by a sign (as replacing an angle with its supplement changes the sign of the cosine); the formulas, therefore, are only "unambiguously correct" in absolute value. Note that we do have this unambiguous corollary: *The pseudofaces of an equihedral tetrahedron —ie, when $W = X = Y = Z$— are mutually perpendicular.*

Finally, what good are cosines without a corresponding Law?[1]

$$\begin{aligned} J^2 + K^2 - 2JK\cos\theta_H &= (Z^2 + X^2 - 2ZX\cos B) + (X^2 + Y^2 - 2XY\cos C) \\ &\quad - \big((Y^2 + Z^2) - (W^2 + X^2) \big) \\ &= \cdots \\ &= W^2 + X^2 + 2WX\cos D \\ K^2 + H^2 - 2KH\cos\theta_J &= W^2 + Y^2 + 2WY\cos E \\ H^2 + J^2 - 2HJ\cos\theta_K &= W^2 + Z^2 + 2WZ\cos F \end{aligned}$$

Note that

$$\begin{aligned} J^2 + K^2 - 2JK\cos(\text{supplement of } \theta_H) &= Y^2 + Z^2 + 2YZ\cos A \\ K^2 + H^2 - 2KH\cos(\text{supplement of } \theta_J) &= Z^2 + X^2 + 2ZX\cos B \\ H^2 + J^2 - 2HJ\cos(\text{supplement of } \theta_K) &= X^2 + Y^2 + 2XY\cos C \end{aligned}$$

We leave the reader to investigate "second-pseudofaces" H_2, J_2, K_2, whose areas satisfy, say,

$$H_2^2 := J^2 + K^2 - 2JK\cos\theta_H$$

although we will point out the geometric interpretations of such elements. They are the pseudofaces of the "exterior" tetrahedra to the given tetrahedron: Reflect point C in segment OA to get C' and a new tetrahedron $OABC'$ that shares facce $\triangle OAB$ with the original. The faces $\triangle OAC'$ (which is congruent to $\triangle OAC$) and $\triangle OAB$ enclose the supplement of the angle bounded by $\triangle OAC$ and $\triangle OAB$, hence the pseudoface corresponding to OA and BC' has area given by

$$Y^2 + Z^2 - 2YZ\cos(\text{supp. of } A) = Y^2 + Z^2 + 2YZ\cos A = H_2^2$$

Whether these exterior tetrahedra attach to edges surrounding a vertex (O) or to edges surrounding a face ($\triangle ABC$) depends upon the preferred definition of the pseudoface angles.

Another Volume Formula. Making appropriate substitutions into the hedronometric volume formula (6), we can achieve this counterpart:

$$81V^4 = H^2J^2K^2\left(1 - 2\cos\theta_H\cos\theta_J\cos\theta_K - \cos^2\theta_H - \cos^2\theta_J - \cos^2\theta_K\right)$$

(The $2\cos\theta_H\cos\theta_J\cos\theta_K$ term is subject to sign change under the alternate definition of the angles involved.)

An equihedral tetrahedron's volume, therefore, (unambiguously) satisfies

$$9V^2 = HJK$$

As a sanity check: A regular tetrahedron with edge-length s has pseudofaces in the form of squares of edge-length $s/\sqrt{2}$ and area $s^2/2$. Thus, the tetrahedron's volume is $\sqrt{s^6/72} = s^3\sqrt{2}/12$, agreeing with a standard formula.

[1] 2022 revision: The 2005 version had reversed the cosines' signs in these and the following formulas.