# A CEVA-LIKE THEOREM FOR TETRAHEDRA

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ABSTRACT. We present a (weak) tetrahedral analogue of Ceva's Theorem for triangles, relating concurrency of lines to signed triple-ratios of areas.

Ceva's Theorem for triangles —described by Grünbaum and Shephard [2] as "among the most attractive and useful results in elementary plane geometry"— can be stated thusly:

**Theorem 1** (Ceva's Theorem)**.** *Given triangle $\triangle ABC$, with lines through $A$, $B$, $C$ meeting opposite edge-lines at respective points $A'$, $B'$, $C'$, the three lines concur if and only if*

$$\left[\frac{BA'}{A'C}\right]\cdot\left[\frac{CB'}{B'A}\right]\cdot\left[\frac{AC'}{C'B}\right] \;=\; 1 \tag{1}$$

*where the factors on the left are* signed ratios *of segment lengths.*[1]

As one might expect, such a straightforward, fundamental result admits a variety of generalizations in different conceptual directions. Remaining in the plane, one can increase figure complexity: for instance, Grünbaum and Shephard [2] consider arbitrary $n$-gonal cycles. Or, one may maintain figure simplicity —*simplex*-ity?— while exploring higher dimensions: Goldberg [1] examines tetrahedra (and "space quadrilaterals"); Landy [3] characterizes concurrence via "vertex mass distributions" of simplices in any-dimensional space.

In this note, we present a new(?) generalization of the second kind; however, while our theorem clearly extends to any-dimensional simplices, in the interest of notational clarity we state and prove it specifically for tetrahedra. As with some other generalizations, we must abandon the satisfyingly-clean "if and only if" nature of Ceva's original result.

**Theorem 2** (Ceva's Theorem for Tetrahedra)**.** *Given tetrahedron $OABC$, with lines through $O$, $A$, $B$, $C$ meeting opposite face-planes at respective points $O'$, $A'$, $B'$, $C'$, the*

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[1]Grünbaum and Shephard [2] define the signed ratio $[WX/YZ]$ for collinear segments as the ratio of lengths $|\overline{WX}|$ and $|\overline{YZ}|$, taken as *positive* (or *negative*) if rays $\overrightarrow{WX}$ and $\overrightarrow{YZ}$ point in *the same direction* (respectively, *opposite directions*). For the purposes of generalization to triple-ratios (see below), we "signify" our ratios component-wise, relative to a given direction along the line: a component is *positive* (or *negative*) when its corresponding ray *matches* (respectively, *opposes*) the line's direction.

*four lines concur only if*

$$
\begin{aligned}
&[\,CBA' : BOA' : OCA'\,] \\
\cdot\,&[\,OAB' : ACB' : COB'\,] \\
\cdot\,&[\,AOC' : OBC' : BAC'\,] \\
\cdot\,&[\,BCO' : CAO' : ABO'\,] \;=\; [\,1:1:1\,]
\end{aligned} \tag{2}
$$

*where the factors on the left are* signed triple-ratios *of triangle areas.*[2,3,4]

*Conversely, if* (2) *holds, and if three of the lines concur, then all four lines concur.*

*Proof.* Concerned only with ratios of coplanar areas, we can apply an affine transformation so as to position $O$ at the origin, with remaining vertices $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$. Letting $P(x,y,z)$ be the point of concurrency for the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$, we deduce

$$A' = \frac{(0,y,z)}{1-x} \qquad B' = \frac{(x,0,z)}{1-y} \qquad C' = \frac{(x,y,0)}{1-z} \tag{3}$$

so that

$$
\begin{aligned}
[\,CBA' : BOA' : OCA'\,] &= \left[\,\frac{1-x-y-z}{2(1-x)} : \frac{z}{2(1-x)} : \frac{y}{2(1-x)}\,\right] \\
&= [\,1-x-y-z : z : y\,]
\end{aligned} \tag{4a}
$$

$$[\,OAB' : ACB' : COB'\,] = [\,z : 1-x-y-z : x\,] \tag{4b}$$

$$[\,AOC' : OBC' : BAC'\,] = [\,y : x : 1-x-y-z\,] \tag{4c}$$

Now, writing $O'$ generically as $(u,v,w)$, with $u+v+w=1$, we compute

$$[\,BCO' : CAO' : ABO'\,] = \left[\,u\,\frac{\sqrt{3}}{2} : v\,\frac{\sqrt{3}}{2} : w\,\frac{\sqrt{3}}{2}\,\right] = [\,u : v : w\,] \tag{5}$$

Then (2) holds if and only if

$$uyz(1-x-y-z) = xvz(1-x-y-z) = xyw(1-x-y-z) \tag{6a}$$

or, equivalently, if and only if

$$\frac{u}{x} = \frac{v}{y} = \frac{w}{z} \tag{6b}$$

which is to say, $O' = \lambda P$ for some $\lambda$: the fourth line, $\overleftrightarrow{OO'}$, contains $P$. □

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[2] As with signed ratios of lengths (see preceding footnote), we "signify" triple-ratios of areas component-wise, relative to a given orientation of the triangle's common plane: component $XYZ$ is *positive* (or *negative*) if the path $X \to Y \to Z \to X$ *matches* (respectively, *opposes*) the plane's orientation. In our coordinatized proof, area-signing issues are automatically accommodated by signs of coordinates.

[3] Note that $[\lambda x : \lambda y : \lambda z] = [x : y : z]$ for any $\lambda \neq 0$, and that triple ratios multiply *component-wise*.

[4] A tip for understanding the order of the sub-triangle components: tint the tetrahedron's opposing edge-pairs *red* ($\overline{OA}$, $\overline{BC}$), *green* ($\overline{OB}$, $\overline{CA}$), *blue* ($\overline{OC}$, $\overline{AB}$), and assign each sub-triangle the color of the edge it abuts; then the components in each triple-ratio can be read simply as [ *red* : *green* : *blue* ].

## Addendum: Whither Menelaus?

Ceva's Theorem is rarely discussed in the absence of a counterpart attributed to Menelaus. (Indeed, Ceva re-discovered the latter and published it simultaneously with his namesake.)

**Theorem 3** (Menelaus' Theorem)**.** *Given triangle $\triangle ABC$, with points $A'$, $B'$, $C'$ on the edge-lines opposite respective vertices $A$, $B$, $C$, the points are collinear if and only if*

$$\left[\frac{BA'}{A'C}\right] \cdot \left[\frac{CB'}{B'A}\right] \cdot \left[\frac{AC'}{C'B}\right] = -1 \tag{7}$$

Grünbaum and Shephard's note [2] is replete with collinearity results for planar $n$-gons; Goldberg instead relates *coplanarity* of points (in the context of space quadrilaterals) to a vanishing product of ratios of lengths. If there is something Menelaus-like to be said about tetrahedra and triple-ratios of areas —What's the "$-1$" version of $[\,1:1:1\,]$, anyway?— it has eluded us, but we include some thoughts here.

Suppose we coordinatize our tetrahedron $OABC$ as previously, and we define these generic points on its face-planes:

$$A'(0, y_A, z_A) \quad B'(x_B, 0, z_B) \quad C'(x_C, y_C, 0) \qquad O'(x_O, y_O, z_O),\ \ x_O + y_O + z_O = 1 \tag{8}$$

Here, the right-hand side of (2) becomes

$$[\, x_O\, y_C\, z_B\, (1 - y_A - z_A) \,:\, x_C\, y_O\, z_A\, (1 - z_B - x_B) \,:\, x_B\, y_A\, z_O\, (1 - x_C - y_C) \,] \tag{9}$$

Now, *collinearity* of the points imposes very strong conditions on the parameters; namely,

$$B' - O' = \beta\left(A' - O'\right) \quad \Longrightarrow \quad \frac{x_B - x_O}{0 - x_O} = \frac{0 - y_O}{y_A - y_O} = \frac{z_B - z_O}{z_A - z_O} \tag{10a}$$

$$C' - O' = \gamma\left(A' - O'\right) \quad \Longrightarrow \quad \frac{x_C - x_O}{0 - x_O} = \frac{y_C - y_O}{y_A - y_O} = \frac{0 - z_O}{z_A - z_O} \tag{10b}$$

so that the degrees of freedom are cut in half, and we have

$$x_B = \frac{x_O\, y_A}{y_A - y_O} \quad z_B = \frac{y_A\, z_O - y_O\, z_A}{y_A - y_O} \quad x_C = \frac{x_O\, z_A}{z_A - z_O} \quad y_C = \frac{y_O\, z_A - y_A\, z_O}{z_A - z_O} \tag{11}$$

whereby (9) reduces proportionally to

$$\left[\, (y_A\, z_O - y_O\, z_A)^2 \,:\, y_A^2\, z_O^2 \,:\, y_O^2\, z_A^2 \,\right] \tag{12}$$

This is a striking simplification, but it's no $[\,1:1:1\,]$. (Interestingly, any expectations of "$-1$" appearing somewhere are *vigorously* dashed by sign-nullifying exponents. Also: the square root of (12) is a triple-ratio such that one component is the sum of the other two.)

On the other hand, *coplanarity* of $O'$, $A'$, $B'$, $C'$ is characterized by this relation:

$$\begin{aligned} 0 &= \left(A' - O'\right) \bullet \left(\left(B' - O'\right) \times \left(C' - O'\right)\right) \\ &= x_O\ (y_A\, z_B + y_C\, z_A - y_C\, z_B) + y_O\,(x_B\, z_A - x_C\, z_A + x_C\, z_B) \\ &\quad + z_O\,(-x_B\, y_A + x_C\, y_A + x_B\, y_C) - x_C\, y_A\, z_B - x_B\, y_C\, z_A \end{aligned} \tag{13}$$

but an appropriate connection to (9) has not yet presented itself.

## References

[1] Goldberg, Nadav. "Spatial Analogues of Ceva's Theorem and its Applications". 2002. http://www.rose-hulman.edu/mathjournal/archives/2002/vol3-n2/goldberg/goldberg.doc

[2] Grünbaum, Branko, and Shephard, G.C. "Ceva, Menelaus, and the Area Principle". *Mathematics Magazine*, Vol. 68, No. 4 (Oct., 1995), pp. 254-268. http://www.jstor.org/stable/2690569

[3] Landy, Steven. "A Generalization of Ceva's Theorem to Higher Dimensions". *The American Mathematical Monthly*, Vol. 95, No. 10 (Dec., 1988), pp. 936-939. http://www.jstor.org/stable/2322390