A HEDRONOMETRIC THEOREM OF MENGER

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Abstract. A theorem of Menger outlines necessary and sufficient conditions for a set of candidate edge-lengths to determine a non-degenerate tetrahedron. This note offers a *hedronometric* counterpart, placing conditions on candidate face- and pseudoface-areas.

Karl Menger [1] proved that the question of whether six positive numbers can serve as the edge-lengths a non-degenerate tetrahedron amounts to the simplest of sanity checks; or, rather, *reality* checks: *The ostensible figure's four face-areas, and its volume, must be* positive real(!) numbers*.* The checks involve Heron's formula for the area of a triangle ((1) and (2)), and the Cayley-Menger formula the volume of a tetrahedron (3).

Theorem 1 (Menger's Theorem)**.** *An ordered sextuple of positive numbers* (a, b, c, d, e, f) *determines a non-degenerate tetrahedron —with concurrent edges of lengths* a*,* b*,* c *opposite edges of respective lengths* d*,* e*,* f *— if and only if the following hold:*

(a) *The Heronic products* $[d, b, c]$*,* $[a, e, c]$*,* $[a, b, f]$*,* $[d, e, f]$ *are positive.*
(b) *The Cayley-Menger determinant* $\langle a, b, c, d, e, f\rangle$ *is positive.*

Here, the Heronic product[1] is defined by

$$[x, y, z] := (x + y + z)\,(-x + y + z)\,(x - y + z)\,(x + y - z) \tag{1}$$

Each of its last three factors encodes an aspect Triangle Inequality —*at most one* of which can fail for given edge-lengths x, y, z— in such a way that the product is positive only when the edge-lengths form a non-degenerate triangle. The first factor affects the product's *sign* not-at-all, while conveniently promoting its *value* to the square of a triangle's area:

$$[x, y, z] \;=\; 16\,(\text{ area of triangle with edge-lengths } x,\, y,\, z\ )^2 \tag{2}$$

That is, a positive Heronic product allows us to infer a *positive real* value for the area of the ostensible triangle, whereas a negative product precludes "real" and a zero product precludes "positive".

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[1]Named for Heron of Alexandria and his formula (*Metrica*, CE 60). See Equation (2).

In a like manner, the Cayley-Menger determinant tests the viability of an ostensible tetrahedron's volume.

(3)
$$\langle a,b,c,d,e,f\rangle := \begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & 0 & f^2 & e^2 & 1 \\ b^2 & f^2 & 0 & d^2 & 1 \\ c^2 & e^2 & d^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 32\cdot 9 \left(\begin{array}{c} \text{volume of tetrahedron} \\ \text{with edge-lengths } a,\ b,\ c,\ d,\ e,\ f \end{array} \right)^2$$

We can infer a *positive real* volume only when the determinant itself is positive.

Thus, given prospective edge-lengths (a, b, c, d, e, f) and these relations

(4)
$$[d,e,f] = 16W^2 \quad [d,b,c] = 16X^2 \quad [a,e,c] = 16Y^2 \quad [a,b,f] = 16Z^2$$
$$\langle a,b,c,d,e,f\rangle = 32\cdot 9V^2$$

a non-degenerate tetrahedron arises, with face-areas W, X, Y, Z, and volume V, if and only if we can take those measures to be positive. Our hedronometric counterpart of Menger's Theorem has a similar character.

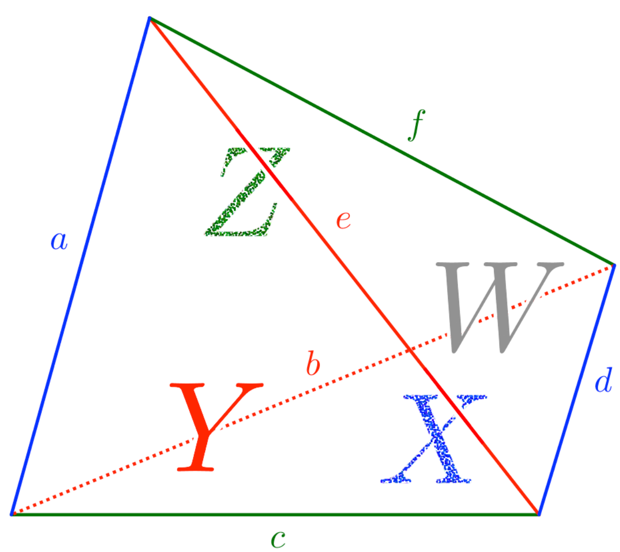

FIGURE 1. A tetrahedron with edges a, b, c, d, e, f and faces W, X, Y, Z. Pseudofaces (not shown) are associated with pairs of opposite edges: H with $\{a,d\}$, J with $\{b,e\}$, K with $\{c,f\}$.

Theorem 2 (Hedronometric Menger's Theorem)**.** *An ordered sextuple of positive numbers* $(X, Y, Z; H, J, K)$ *determines a non-degenerate tetrahedron —with faces of area* X, Y, Z, *and with pseudofaces*[2] *of area* H, J, K *associated with respective face-pairs*[3] $\{Y, Z\}$, $\{Z, X\}$, $\{X, Y\}$*— if and only if the following hold:*

(a) $H^2 + J^2 + K^2 - X^2 - Y^2 - Z^2$ *is positive.*
(b) *The Heronic products* $[H, Y, Z]$, $[X, J, Z]$, $[X, Y, K]$ *are positive.*
(c) *The determinant* $\langle X, Y, Z; H, J, K\rangle$ *is positive, where*

$$\langle X, Y, Z; H, J, K\rangle := \begin{vmatrix} 2X^2 & K^2 - X^2 - Y^2 & J^2 - Z^2 - X^2 \\ K^2 - X^2 - Y^2 & 2Y^2 & H^2 - Y^2 - Z^2 \\ J^2 - Z^2 - X^2 & H^2 - Y^2 - Z^2 & 2Z^2 \end{vmatrix} \tag{5}$$

Each condition of Theorem 2 is its own "reality check" on a fundamental hedronometric property. Condition (a) is required by the Sum of Squares identity involving the tetrahedron's fourth (presumably non-degenerate) face-area, W:

$$W^2 + X^2 + Y^2 + Z^2 = H^2 + J^2 + K^2 \quad \Longrightarrow \quad H^2 + J^2 + K^2 - X^2 - Y^2 - Z^2 = W^2 > 0 \tag{6}$$

The products in (b) correspond to instances of the tetrahedral Law of Cosines; for example, with A the dihedral angle between faces of area Y and Z,

$$H^2 = Y^2 + Z^2 - 2YZ\cos A \quad \Longrightarrow \quad [H, Y, Z] = 4\, Y^2 Z^2\, \sin^2 A > 0 \tag{7}$$

Finally, the determinant in (c), upon replacing H^2, J^2, K^2 with expressions in cosines of appropriate dihedral angles A, B, C, reduces to the fourth power of the figure's volume:

$$8\, X^2 Y^2 Z^2\, (1 - 2\cos A\cos B\cos C - \cos^2 A - \cos^2 B - \cos^2 C) = 2 \cdot 81\, V^4 > 0 \tag{8}$$

A combination of the above yields another set of relations worth mentioning:

$$[H, Y, Z] = 9V^2 a^2 \qquad [X, J, Z] = 9V^2 b^2 \qquad [X, Y, K] = 9V^2 c^2 \tag{9a}$$

With these, we see that (a), (b), (c) are "reality checks" on the viability of consequent edge-lengths a, b, c. Of course, we also have, with $W := \sqrt{H^2 + J^2 + K^2 - X^2 - Y^2 - Z^2}$,

$$[H, W, X] = 9V^2 d^2 \qquad [J, W, Y] = 9V^2 e^2 \qquad [K, W, Z] = 9V^2 f^2 \tag{9b}$$

as necessary relations. If we can show that conditions (a), (b), (c) imply that these Heronic products are positive (so that d, e, f can themselves be taken as positive), our theorem is proven, since one readily verifies that the consequent edge-lengths a, b, c, d, e, f satisfy Menger's Theorem.

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[2] Geometrically, a "pseudoface" is the quadrilateral projection of a tetrahedron into a plane parallel to a pair of opposite edges. Algebraically, we can formally define a "pseudoface area" via edge-lengths; e.g.,

$$16H^2 = 4a^2d^2 - \left(b^2 - c^2 + e^2 - f^2\right)^2$$

This amounts to Bretschneider's Formula for the area of the pseudoface for opposite edges a and d, since the quadrilateral projection has edge-lengths $\sqrt{b^2 - h^2}$, $\sqrt{c^2 - h^2}$, $\sqrt{e^2 - h^2}$, $\sqrt{f^2 - h^2}$, where h is the distance between lines containing edges a and d.

[3] E.g., pseudoface H corresponds to opposite edges a and d, and a is the edge common to Y and Z.

Observe that the positivity of $[H,Y,Z]$ implies $(Y-Z)^2 < H^2 < (Y+Z)^2$, which in turn guarantees the existence of an A, strictly between 0 and π, satisfying (7). Likewise, we have some B and C, so that $\langle X,Y,Z;H,J,K\rangle$ reduces to the form in (8). The assumed positivity of the determinant then implies

$$\sin B \sin C > |\cos A + \cos B \cos C| \tag{10}$$

On the other hand, expanding $[H,W,X]$ similarly, gives

$$\begin{aligned} [H,W,X] &= 4X^2 \left(Y^2 \sin^2 C + Z^2 \sin^2 B - 2YZ\,(\cos A + \cos B \cos C) \right) \\ &> 4X^2 \left(Y^2 \sin^2 C + Z^2 \sin^2 B - 2YZ \sin B \sin C \right) \\ &= 4X^2 \left(Y \sin C - Z \sin B \right)^2 \\ &\geq 0 \end{aligned} \tag{11}$$

In the same way, we assure $[J,W,Y]$ and $[K,W,Z]$ are positive, proving the theorem. $\square$

Theorem 2 has an unsettling lack of symmetry, so we offer a variant as a corollary:

Corollary (Hedronometric Menger's Theorem, Symmetric Form)**.** *An ordered septuple of positive numbers $(W,X,Y,Z;H,J,K)$ determines a non-degenerate tetrahedron —with face-areas W, X, Y, Z, and pseudoface-areas H, J, K corresponding to respective face-pair pairs $\{W,X;Y,Z\}$, $\{W,Y;Z,X\}$, $\{W,Z;X,Y\}$— if and only if the following hold:*

(a) $W^2 + X^2 + Y^2 + Z^2 = H^2 + J^2 + K^2$
(b) *The Heronic products $[H,Y,Z]$, $[X,J,Z]$, $[X,Y,K]$, $[H,W,X]$, $[J,W,Y]$, $[K,W,Z]$ are positive. (For sufficiency, pick three: one with H, one with J, one with K.)*
(c) *The determinant $\langle W,X,Y,Z;H,J,K\rangle$ is* negative, *where*[4]

$$\langle W,X,Y,Z;H,J,K\rangle := \begin{vmatrix} 4W^2 & H^2 & J^2 & K^2 & 1 \\ H^2 & 4X^2 & K^2 & J^2 & 1 \\ J^2 & K^2 & 4Y^2 & H^2 & 1 \\ K^2 & J^2 & H^2 & 4Z^2 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \bmod \begin{pmatrix} W^2+X^2+Y^2+Z^2 \\ -H^2-J^2-K^2 \end{pmatrix}$$
$$= -32 \begin{pmatrix} 2W^2X^2Y^2 + 2W^2Y^2Z^2 + 2W^2Z^2X^2 + 2X^2Y^2Z^2 + H^2J^2K^2 \\ -H^2\left(W^2X^2+Y^2Z^2\right) - J^2\left(W^2Y^2+Z^2X^2\right) - K^2\left(W^2Z^2+X^2Y^2\right) \end{pmatrix} \tag{12}$$

For proof, we need only mention how the determinant in (c), with the substitution $W^2 = H^2 + J^2 + K^2 - X^2 - Y^2 - Z^2$, relates to that in the Theorem:

$$\langle W,X,Y,Z;H,J,K\rangle = -16\,\langle X,Y,Z;H,J,K\rangle = -32 \cdot 81 V^4 \tag{13}$$

References

[1] Menger, K. Untersuchungen über allgemeine Metrik. *Math. Ann.* **100** (1928), pp. 75–163 (in particular, pp. 133–136).

[4] Interestingly, the *modulo* eliminates exactly $-4\,[H,J,K]\,(W^2+X^2+Y^2+Z^2-H^2-J^2-K^2)$.