

**(ALMOST) EVERYTHING YOU NEED TO REMEMBER ABOUT
TRIGONOMETRY, IN ONE SIMPLE DIAGRAM
(DRAFT)**

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ABSTRACT. The “Complete Triangle” figure serves as a treasure-trove of trigonometric facts, incorporating a large number of concepts that students are expected to remember about a Trigonometry course, from the computational to the terminological.

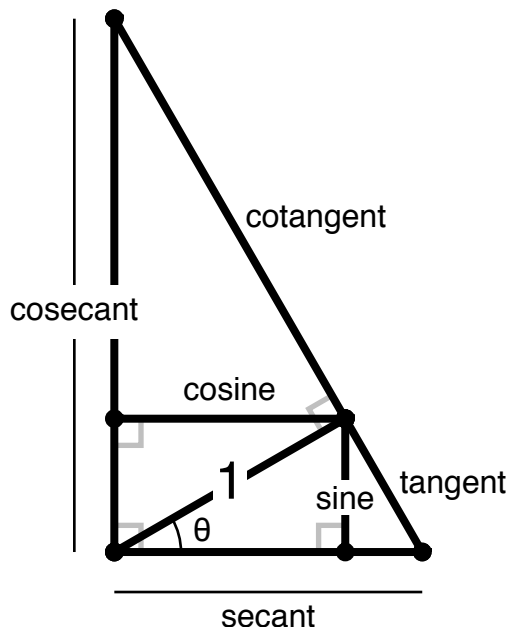


FIGURE 1. The Complete Triangle.

1. INTRODUCTION

Trigonometry —literally the study of “triangle measurement”— is, more fundamentally, the study of *measurement*. In particular, it bridges the gap between two wildly different notions of measurement: the *lengths of segments* and the *sizes of angles*.

As we all know, an angle is defined as being bounded by two infinitely long rays emanating from a common point, the vertex (V). (See Figure 2.) If we were to locate two points (A and B), one on each ray, the same distance from the vertex, our Euclidean instincts might compel us to draw the line containing them; here, we’ll make do with a segment. We might also draw a circle with center V passing through A and B , recognizing the segment as a *chord* of the circle.

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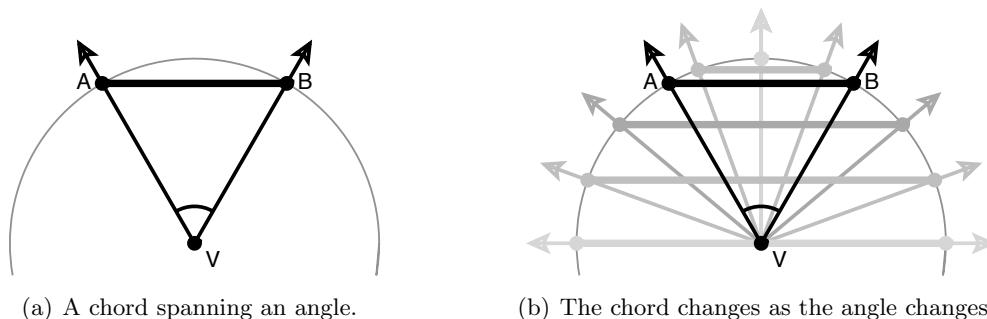


FIGURE 2. A chord associated with an angle.

The connection between the chord and the angle —between the *length* of the chord and the *size* of the angle— is clearly a tight one. (Provided that we’re always talking about A and B being the same chosen distance from V .) An angle with the smallest possible size (0°) would yield a chord with the shortest possible length (0); an angle of the largest possible size¹ (180°) would yield the longest possible chord (with length equal to the diameter of the circle, which is twice the chosen distance from A or B to V); somewhere in there is a chord tied to the right angle (90°), or to any angle you prefer. (See Figure 2(b).)

Note that, while the *qualitative* connection is clear —the wider the angle, the longer the chord— the *quantitative* connection is extremely muddy. The chord that “goes with” the 180° angle is not twice as long as the chord that goes with the 90° angle; it’s twice as long as the chord that goes with the 60° angle! Part of trigonometry is understanding the seemingly bizarre nature of that connection,² but the point here is that the connection *exists*: If you know the size of the angle, then you can get the length of the chord, and vice-versa. The connection makes the information virtually interchangeable, which gives us problem-solving options.

Mathematicians never being content to leave well enough alone, trigonometric tradition has handed down, not one, but *six*³ segments that “go with” an angle (none of which is the chord described above, although one is related to it). Each segment its own special connection to the angle and offers its own special problem-solving advantage (not to mention: disadvantages). These segments —more precisely, their lengths— give rise to what we know as the standard trig functions: sine, tangent, secant, cosine, cotangent, and cosecant. Sadly, many (if not most) modern treatments of trigonometry ignore almost all of the underlying segments, robbing the functions of their geometry and their *meaning*, and, thus, making the subject appear arbitrary and cryptic to students. This paper seeks to restore the segments to prominence, combining them into a remarkable figure called the “Complete Triangle” that promotes *comprehension* (not merely *memorization*) of many fundamental principles of trigonometry.

2. THE COMPLETE TRIANGLE

We’ll adopt a conventional setting by starting with the unit circle in the coordinate plane. (See Figure 3.) That is, consider a circle of radius 1, centered at the origin (O). For the time being, we’ll

¹For the purposes of this discussion, angles don’t get bigger than a straight angle.

²And that’s *not easy*! Although the connection had been studied and used for thousands of years, the *formula* for converting angle sizes into chord lengths wasn’t known until Madhava at the turn of the 1400s in the East or Isaac Newton in the 1670s in the West. (How do I cite the on-line *Encyclopedia Britannica*?) Before then, the state of the art was effectively to compile look-up tables of values, a practice that hand-held calculators made obsolete only a few decades ago.

³Actually, there were more, but they have essentially fallen out of favor.

restrict our attention to the first quadrant, letting X and Y be the points where the circle crosses the x - and y -axes, and choosing P to be a point somewhere in the quarter-circular arc between them. The angle XOP , which we'll say has measure θ , is the focus of our attention.

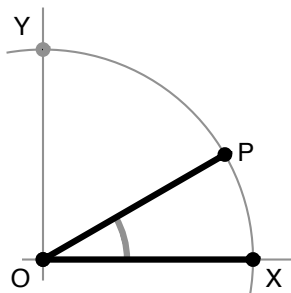


FIGURE 3. A first quadrant angle.

The first of our segments that “go with” θ (Figure 4(a)) is the perpendicular dropped from P to the x -axis. Its seemingly unmotivated construction traces back to the segment of the Introduction: if we mirror the angle POX in the x -axis, then we see this segment as *half* of the chord spanning *twice* the angle θ . Over the years, this semi-chord—which we now know as the *sine* segment—stole the mathematical spotlight from the full chord.

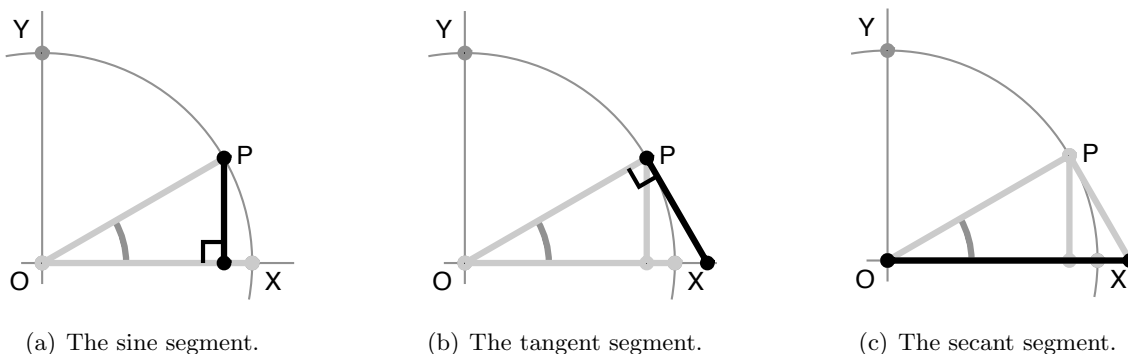


FIGURE 4. Three segments associated with the angle.

The second segment (Figure 4(b)) is the perpendicular “raised” from P to the x -axis. Being at right angles to a radius of the circle, this segment is tangent to the circle, earning the name of *tangent* segment.⁴

The third segment (Figure 4(c)) has already been determined: It joins the origin to the point where the tangent segment meets the x -axis. As part of a line passing through the circle, it receives the name *secant* segment.⁵

The lengths of these segments are as intimately connected to the size of the angle as the chord discussed in the Introduction. Again, the exact computational connection is tricky (and beside the

⁴Alternatively, we could have defined this segment by raising a perpendicular from X to the extended radius OP . Doing this, and mirroring POX as we did with sine, would reveal the segment to be *half* the tangent spanning *twice* the angle θ in a natural way.

⁵If we think of this segment as being *half* of some other (an insight useful for sine and tangent), then that other would be a proper secant segment, passing all the way through the circle.

point here), but the general sense is clear: bigger angles have longer segments, and to know the angle's size is to know each segment's length (and vice-versa). We'll cover specific observations about these connections in later sections.

The final three segments (Figure 5) form a perfect complement to the first three, because they're constructed in the same way, but relative to the angle POY , the *complement* of POX : the perpendicular dropped from P to the y -axis is the *complementary sine*—that is, the *co-sine*—segment; the perpendicular raised from P to the y -axis is the *complementary tangent* (*co-tangent*) segment; and the *complementary secant* (*co-secant*) segment joins origin to the intersection of the co-tangent segment and the y -axis.

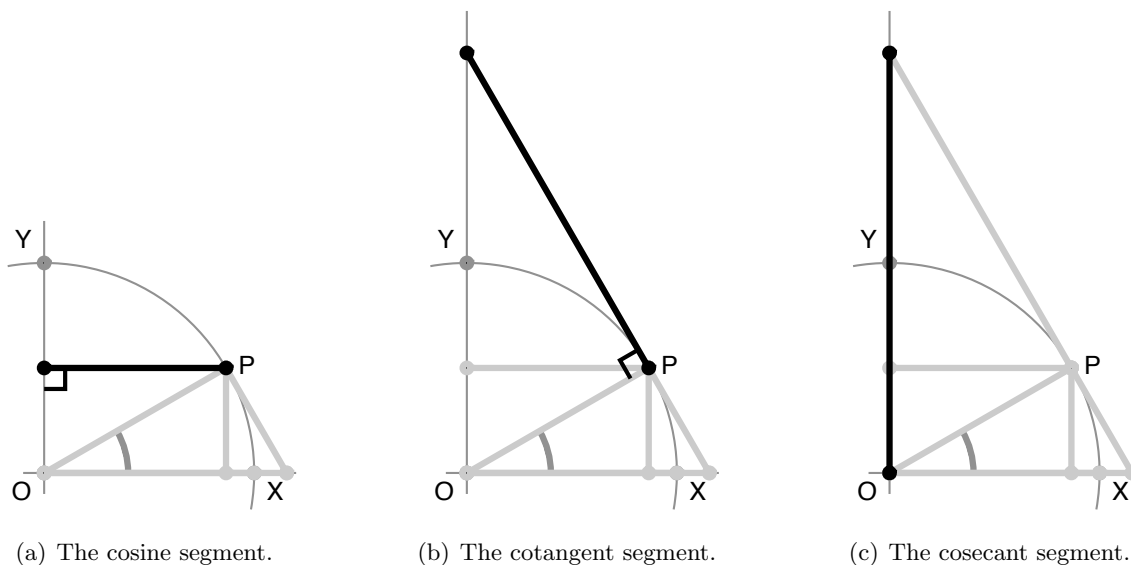


FIGURE 5. Three co-segments associated with the angle.

Together, these six segments, along with the radius segment OP of length 1, form the figure we call the “Complete Triangle” (Figure 1). The geometry of the figure neatly encodes numerous properties of, and relationships among, the trigonometric functions that take their names from the segments.

NOTE: As we explore these properties and relationships, we will continue to restrict our attention to the first quadrant.

3. ETYMOLOGICAL-DEFINITIONAL PROPERTIES

Identifying the trig *functions* with the trig *segments* in the Complete Triangle makes their names obvious:

- The tangent and cotangent segments are parts of a *tangent line* of the circle.
- The secant and cosecant segments are parts of *secant lines* of the circle.
- The sine and cosine segments are *semi-chords* of the circle.⁶

⁶Blame ancient transcribers for the fact that this statement isn't as satisfyingly obvious as the others:

Moreover, the “co-trig” segments being constructed in relation to the *complement* of the angle θ —which we might call “ $\text{co}\theta$ ”—makes that terminology (and some fundamental equalities) obvious as well:

- The cotangent segment for θ is the tangent segment for $\text{co}\theta$. ($\cot(\theta) = \tan(\text{co}\theta)$)
- The cosecant segment for θ is the secant segment for $\text{co}\theta$. ($\csc(\theta) = \sec(\text{co}\theta)$)
- The cosine segment for θ is the sine segment for $\text{co}\theta$. ($\cos(\theta) = \sin(\text{co}\theta)$)
- (Also, the radius segment for θ is the radius segment for $\text{co}\theta$. In many ways, the radius serves as its own co-segment.)

We see, then, that the names for the trig functions actually *make sense*. This alone secures the Complete Triangle’s position as an important pedagogical tool. Textbooks that introduce the trig functions as ratios of lengths in a right triangle (see Sections 5.3 and 5.4) impede learning by reducing the associated vocabulary to wholly-unmotivated “buzzwords”.

4. DYNAMIC PROPERTIES

Intuition-building insights regarding the interrelation of the trigonometric values can be gleaned from treating the Complete Triangle as a *dynamic* figure, sweeping the radius segment through the entire quarter circle, from $\theta = 0^\circ$ to $\theta = 90^\circ$.

4.1. Fun-House Mirror Properties. Notice that the radius segment (“1”) neatly separates the “ordinary elements” (angle θ and the segments \tan , \sec , and \sin) from the “co-elements” (angle $\text{co}\theta$ and the segments \cot , \csc , and \cos). In a way, the radius acts as a fun-house mirror through which each element can see its co-element as a version of itself, distorted in such a way that a co-element grows as the element shrinks, and vice versa.

| | | | |
|-------------|------------------------|-------------------|------------------------|
| As θ | gets bigger (smaller), | $\text{co}\theta$ | gets smaller (bigger). |
| As \tan | gets bigger (smaller), | \cot | gets smaller (bigger). |
| As \sec | gets bigger (smaller), | \csc | gets smaller (bigger). |
| As \sin | gets bigger (smaller), | \cos | gets smaller (bigger). |

Important: This observation is *qualitative*, not *quantitative*. It does not precisely describe *how much* a co-element grows in relation to *how much* an element shrinks. In fact, the formulas governing the relative growth/shrinkage factors are different for each element/co-element pair.

Note that every element exactly *matches* its co-element when $\theta = \text{co}\theta = 45^\circ$. As a consequence of the Fun House Mirror Properties, we have that, if $\theta < \text{co}\theta$, then each of the ordinary segments are smaller than its co-uterpart; and, if $\theta > \text{co}\theta$, then each ordinary segment is larger than its co-uterpart. Noting that θ and $\text{co}\theta$ count as co-elements, we can unify these statements into one:

Our modern word *sine* is derived from the Latin word *sinus*, which means “bay” or “fold”, from a mistranslation (via Arabic) of the Sanskrit word *jiva*, alternatively called *jya*. Aryabhata used the term *ardha-jiva* (“half-chord”), which was shortened to *jiva* and then transliterated by the Arabs as *jiba*. European translators like Robert of Chester and Gherardo of Cremona in 12th-century Toledo confused *jiba* for *jaib*, meaning “bay”, probably because *jiba* and *jaib* are written the same in the Arabic script (this writing system, in one of its forms, does not provide the reader with complete information about the vowels). [History of trigonometric functions. (2006, November 5). In Wikipedia, The Free Encyclopedia. Retrieved 03:48, December 22, 2006, from http://en.wikipedia.org/w/index.php?title=History_of_Trigonometric_functions&oldid=85828396]

All ordinary elements are either simultaneously larger than, simultaneously equal to, or simultaneously smaller than their respective co-elements.

4.2. Range Properties. At the extremes of the radius segment’s sweep, we find that the some trig segments collapse into points, others coincide with the radius segment, and yet others become infinitely long *rays* parallel to the x - or y -axis. We read from these facts the ranges of the corresponding trigonometric functions.

| | 0° | 90° | Range ⁷ |
|-----------|------------------------|------------------------|--------------------|
| sine | collapsed to point | coincident with radius | 0 to 1 |
| tangent | collapsed to point | infinitely long | 0 to ∞ |
| secant | coincident with radius | infinitely long | 1 to ∞ |
| | | | |
| cosine | coincident with radius | collapsed to point | 1 to 0 |
| cotangent | infinitely long | collapsed to point | ∞ to 0 |
| cosecant | infinitely long | coincident with radius | ∞ to 1 |

Disclaimer about ∞ : The table reflects ranges as perceived when restricting our attention to first quadrant angles. As segments such as the tangent segment approach ray status, their lengths reach infinitely large —and unambiguously *positive*— values. As we discuss later, when we consider angles beyond the first quadrant, the *signed* length of the tangent segment at 90° cannot be unambiguously identified, so that the trigonometric value $\tan 90^\circ$ is *officially* considered “undefined”.

Note that at $\theta = 45^\circ$ the Complete Triangle is half of a square whose center is the point where the radius, tangent, and cotangent segments meet. These three segments are clearly congruent, so that $\tan 45^\circ = \cot 45^\circ = 1$.

5. “GENERAL POSITION” PROPERTIES (IDENTITIES AND COMPARATIES)

The extreme (or “degenerate”) cases —namely, $\theta = 0^\circ$ and $\theta = 90^\circ$ — cause elements of the Complete Triangle to collapse into each other or extend indefinitely. If we set aside these cases for a moment, we see that the Complete Triangle figure in *general position* contains seven similar right triangles, *regardless of the angle involved*. Leveraging geometric theorems regarding right triangles and families of similar triangles, we can draw a number of computational conclusions —equalities and comparisons— that themselves hold *regardless of the angle involved*. Minor tweaks usually suffice to extend these results to apply to degenerate cases. The term for such universal equalities is “identities”; with (so far as I know) no official term for the analogous notion of universal comparisons, we’ll call them “comparaties”.

To get at our results in this section, we name the triangles in a way that reflects co-triangle pairs, with ΔIV is it own co-triangle. In each triangle, we refer to the leg *opposite* the angle congruent to θ as *opp*, the leg *adjacent* to the angle congruent to θ as *adj*, and the hypotenuse as *hyp*.

Of course, ΔI and $\Delta \text{co-I}$ are congruent, so we don’t need to consider them separately.

5.1. Comparaties. The *Triangle Inequality* is an elementary expression of the concept that the shortest distance between two points is a straight line.

The sum of the lengths of two sides of any triangle is greater than (or, in degenerate cases, equal to) the length of the third side.

| Δ | <i>hyp</i> | <i>opp</i> | <i>adj</i> |
|----------|------------|------------|------------|
| I | 1 | sin | cos |
| co-I | 1 | sin | cos |
| II | sec | tan | 1 |
| co-II | csc | 1 | cot |
| III | tan | sec - cos | sin |
| co-III | cot | cos | csc - sin |
| IV | tan + cot | sec | csc |

For our triangles, the result implies

$$opp + hyp \geq adj \quad adj + hyp \geq opp \quad opp + adj \geq hyp$$

However, since our triangle are *right* triangles, we can exploit this fact

The hypotenuse is the longest side of a right triangle.

to refine our list of comparisons to

$$hyp \geq adj \quad hyp \geq opp \quad opp + adj \geq hyp$$

with equality only in degenerate cases

Table 1 lists the comparaties that arise from applying these facts to our seven triangles.

| | | | | |
|-------------------|-----------------|--------|-----------|--|
| Δ I & co-I | 1 | \geq | cos | (= only at 0°) |
| | 1 | \geq | sin | (= only at 90°) |
| | sin + cos | \geq | 1 | (= only at 0° and 90°) |
| Δ II | sec | \geq | 1 | (= only at 0°) |
| | sec | \geq | tan | (= only at 90°) |
| | tan + 1 | \geq | sec | (= only at 0° and 90°) |
| Δ co-II | csc | \geq | cot | (= only at 0°) |
| | csc | \geq | 1 | (= only at 90°) |
| | 1 + cot | \geq | csc | (= only at 0° and 90°) |
| Δ III | tan | \geq | sin | (= only at 0°) |
| | tan | \geq | sec - cos | (= only at 90°) |
| | sec - cos + sin | \geq | cot | (= only at 0° and 90°) |
| Δ co-III | cot | \geq | csc - sin | (= only at 0°) |
| | cot | \geq | cos | (= only at 90°) |
| | csc - sin + cos | \geq | cot | (= only at 0° and 90°) |
| Δ IV | tan + cot | \geq | csc | (= only at 0°) |
| | tan + cot | \geq | sec | (= only at 90°) |
| | sec + csc | \geq | tan + cot | (= only at 0° and 90°) |

TABLE 1. Seventeen Comparaties. (The last is redundant.)

Apart from the comparaties bounding trig values above or below by 1, these results get tend to get very little attention. Even so, some of them help reinforce observations one makes about the graphs of the trigonometric functions.

- $\tan \geq \sin$. We notice that the graphs coincide at a point (the origin), and that the graph of tan rides *just above* that of sin for a while before they split dramatically. Likewise for

$\cot \geq \cos$, although the point of coincidence is different.

- $\sec \geq \tan$. Here we note that, although both graphs are asymptotic as θ approaches 90° , the graph of \sec is always dominant. In light of the comparaty $\tan + 1 \geq \sec$, we see that the dominance is slight at best, with the secant graph sandwiched between the tangent graph and a one-vertical-unit translate of the tangent graph. Likewise for $\csc \geq \cot$.

The comparaties involving sums of functions may provide their own insights in some context or other.

5.2. Pythagorean Identities. Perhaps the most well-known result in all of geometry is the Theorem of Pythagoras, which states

The sum of the squares of the lengths of the two legs of a right triangle is equal to the square of the length of the hypotenuse.

$$opp^2 + adj^2 = hyp^2$$

Table 2 applies this theorem to our seven triangles.

| | | | |
|------------|----------------------------|---|-------------------|
| Δ I & co-I | $\cos^2 + \sin^2$ | = | 1 |
| Δ II | $\sin^2 + (\sec - \cos)^2$ | = | \tan^2 |
| Δ co-II | $\cos^2 + (\csc - \sin)^2$ | = | \cot^2 |
| Δ III | $1 + \tan^2$ | = | \sec^2 |
| Δ co-III | $1 + \cot^2$ | = | \csc^2 |
| Δ IV | $\sec^2 + \csc^2$ | = | $(\tan + \cot)^2$ |

TABLE 2. Pythagorean Identities

Of course, the identities involving 1 are by far the most significant —they make up part of that traditional “core set”— but the others make interesting curiosities.

5.3. Internal Proportionality Properties. A family of similar triangles admits the following (*internal*) proportionality property:

The ratio of the lengths of two sides of a given triangle is equal to the ratio of the lengths of the corresponding sides of any triangle similar to the given one.

We can put this another way:

Each of the following ratios is constant for all triangles in the Complete Triangle figure:

$$opp/hyp \qquad adj/hyp \qquad opp/adj$$

Table 3 gives the full list equated ratios, which (by equating pairs) make for 45 individual proportionality identities (or 90, if you count reciprocated versions).

Amid the chaos are four special relations:

$$\frac{\sec}{1} = \frac{1}{\cos} \qquad \frac{\csc}{1} = \frac{1}{\sin} \qquad \frac{\tan}{1} = \frac{\sin}{\cos} \qquad \frac{\cot}{1} = \frac{\cos}{\sin} \left(= \frac{1}{\tan} \right)$$

In most modern textbook treatments of trigonometry, these relations are used as (wholly unmotivated) *definitions* of the \sec , \csc , \tan , and \cot values instead of as *theorems* regarding those values. Such treatments not only rob the words of their inherent geometrical meaning, they convey

| | Δ I & co-I | Δ II | Δ co-II | Δ III | Δ co-III | Δ IV |
|---------------------------------|---------------------|-----------------------|-----------------------|------------------------------|------------------------------|------------------------------|
| $\frac{\text{opp}}{\text{hyp}}$ | $\frac{\sin}{1}$ | $= \frac{\tan}{\sec}$ | $= \frac{1}{\csc}$ | $= \frac{\sec - \cos}{\tan}$ | $= \frac{\cos}{\cot}$ | $= \frac{\sec}{\tan + \cot}$ |
| $\frac{\text{adj}}{\text{hyp}}$ | $\frac{\cos}{1}$ | $= \frac{1}{\sec}$ | $= \frac{\cot}{\csc}$ | $= \frac{\sin}{\tan}$ | $= \frac{\csc - \sin}{\cot}$ | $= \frac{\csc}{\tan + \cot}$ |
| $\frac{\text{opp}}{\text{adj}}$ | $\frac{\sin}{\cos}$ | $= \frac{\tan}{1}$ | $= \frac{1}{\cot}$ | $= \frac{\sec - \cos}{\sin}$ | $= \frac{\cos}{\csc - \sin}$ | $= \frac{\sec}{\csc}$ |

TABLE 3. Internal Proportionality Identities.

to students the notion that the words of mathematics *don't have* inherent meaning, that math terms are no more than miscellaneous syllables attached at random to some fact or another.

That said, the facts that cosecant is the reciprocal of sine, secant is the reciprocal of cosine, and cotangent is the reciprocal of tangent *are* of considerable significance, such significance that using the facts in a definitional role isn't *unwarranted* (only somewhat pedagogically distasteful). Interestingly, the Complete Triangle figure provides a convenient way to remember which segment is the reciprocal of which:

Segment pairs whose lengths are reciprocals are *parallel* in the Complete Triangle.

$\sin \parallel \csc \quad \cos \parallel \sec \quad \tan \parallel \cot \quad (1 \parallel 1)$

5.4. External Proportionality Properties. A pair of similar triangles exhibit this (*external*) proportionality property:

The ratios of the lengths (a, b, c) of the sides a triangle to the respective lengths (a', b', c') of the corresponding sides of a similar triangle are equal.

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

For our purposes, we can use the restatement:

Writing (hyp, opp, adj) and (hyp', opp', adj') for the lengths of sides of two sub-triangles of the Complete Triangle,

$$\frac{hyp}{hyp'} = \frac{opp}{opp'} = \frac{adj}{adj'}$$

All told, the 45 (or 90, counting reciprocation) individual identities derived from Table 4 duplicate those derived from the internal proportions in Table 3. Even so, the external-vs-internal approaches are subtly —though not-insignificantly— different, warranting separate consideration.

5.5. Area Properties. We can derive a few final identities by computing areas of triangles from the Complete Triangle figure in two different ways ...

- Half the product of the lengths of the legs.
- Half the product of the length of the hypotenuse and the length of the altitude *to* the hypotenuse.

... and equating *double* (to discard the unnecessary “ $\frac{1}{2}$ ”s) the resulting expressions, as shown in Table 5.

These results can be derived by cross-multiplying appropriate proportions from preceding sections, but, again, the distinctive approach is worth highlighting. Besides, the last of these is one of the most interesting of the non-“core” identities.

| | | | | |
|-------------------|---|---|---|---|
| | Δ II | Δ co-II | Δ III | Δ co-III |
| Δ I & co-I | $\frac{1}{\sec} = \frac{\sin}{\tan} = \frac{\cos}{1}$ | $\frac{1}{\csc} = \frac{\sin}{1} = \frac{\cos}{\cot}$ | $\frac{1}{\tan} = \frac{\sin}{\sec - \cos} = \frac{\cos}{\sin}$ | $\frac{1}{\cot} = \frac{\sin}{\cos} = \frac{\cos}{\csc - \sin}$ |
| Δ II | — | $\frac{\sec}{\csc} = \frac{\tan}{1} = \frac{1}{\cot}$ | $\frac{\sec}{\tan} = \frac{1}{\sec - \cos} = \frac{1}{\sin}$ | $\frac{\sec}{\cot} = \frac{\tan}{\cos} = \frac{1}{\csc - \sin}$ |
| Δ co-II | — | — | $\frac{\csc}{\tan} = \frac{1}{\sec - \cos} = \frac{\cot}{\sin}$ | $\frac{\csc}{\cot} = \frac{1}{\cos} = \frac{\cot}{\csc - \sin}$ |
| Δ III | — | — | — | $\frac{\tan}{\cot} = \frac{\sec - \cos}{\cos} = \frac{\sin}{\csc - \sin}$ |
| Δ co-III | — | — | — | — |

| | |
|-------------------|---|
| | Δ IV |
| Δ I & co-I | $\frac{1}{\tan + \cot} = \frac{\sin}{\sec} = \frac{\cos}{\csc}$ |
| Δ II | $\frac{\sec}{\tan + \cot} = \frac{\tan}{\sec} = \frac{1}{\csc}$ |
| Δ co-II | $\frac{\csc}{\tan + \cot} = \frac{1}{\sec} = \frac{\cot}{\csc}$ |
| Δ III | $\frac{\tan}{\tan + \cot} = \frac{\sec - \cos}{\sec} = \frac{\sin}{\csc}$ |
| Δ co-III | $\frac{\cot}{\tan + \cot} = \frac{\cos}{\sec} = \frac{\csc - \sin}{\csc}$ |

TABLE 4. External Proportionality Identities. $\frac{\text{hyp}}{\text{hyp}'} = \frac{\text{opp}}{\text{opp}'} = \frac{\text{adj}}{\text{adj}'}$

| | Legs | Hypotenuse & Altitude |
|----------------|-------------------|---------------------------|
| Δ II | $\tan \cdot 1$ | $= \sec \cdot \sin$ |
| Δ co-II | $\cot \cdot 1$ | $= \csc \cdot \cos$ |
| Δ IV | $\sec \cdot \csc$ | $= (\tan + \cot) \cdot 1$ |

TABLE 5. Area Identities.

6. THE COMPLETE TRIANGLE IN ANY QUADRANT

Here's a quick outline, to be refined in an upcoming draft.

- The structure of the Complete Triangle is built off of an angle's "reference angle" in a quadrant.
 - Sine is a vertical semi-chord, and cosecant is (part of) a vertical secant line.
 - Cosine is a horizontal semi-chord, and cosecant is (part of) a horizontal secant line.
 - Tangent and cotangent are (parts of) the line tangent to the endpoint of the radius segment (with tangent connecting to the secant segment and cotangent connecting to the cosecant segment).
- The values of the trig functions are *signed lengths* of the corresponding segments, according to some seemingly-arbitrary rules (which I'll justify below).
 - Sine and cosecant are *positive* when these segments fall above the x-axis (Quadrants I and II), and *negative* when the segments fall below the x-axis (Quadrants III and IV).
 - Cosine and secant are *positive* when the segments fall to the right of the y-axis (Quadrants I and IV), and *negative* when the segments fall to the left of the y-axis (Quadrants II and III).
 - Tangent and cotangent are *positive* in Quadrants I and III, and *negative* in Quadrants II and IV. (A good exercise for students is to come up with the Quadrant-free counterpart for this rule. Here's a decent attempt: Tangent is *positive* when the segment "points clockwise" from the point of tangency, and *negative* when the segment "points counter-clockwise"; vice-versa for cotangent.)
- The sign assignments make secant and tangent "undefined" at angles coterminal with 90° and 270° ; they make cosecant and cotangent "undefined" at angles coterminal with 0° and 180° . To see why, consider the Complete Triangles associated with angles *extremely close*

to either side of $\theta = 90$. The tangent segment for θ ever-so-slightly-less-than 90° is an extremely large *positive* number; the closer θ gets to 90° , the more extreme the largeness, until $\theta = 90^\circ$ is expected to give a tangent value of ∞ . On the other hand, the tangent segment for θ ever-so-slightly-more-than 90° is an extremely large *negative* number; the closer θ gets to 90° in that case, the more extreme the largeness, until $\theta = 90^\circ$ is expected to give a tangent value of $-\infty$. Because both ∞ and $-\infty$ are equally justifiable, the expression “ $\tan 90^\circ$ ” is (in this larger context) ambiguous, making the value “undefined”.

6.1. Explaining the Signs. I don’t normally introduce the Unit Circle as the vehicle for defining sine and cosine for arbitrary angles. I prefer to build toward a “full circle” (and beyond) view of Trigonometry by establishing a firm foundation of understanding in Quadrant I alone, focussing as much as possible on geometric principles. For one thing, this helps keep Geometry from being “that weird course I took in between Algebra I and Algebra II”. More importantly, it highlights the general notion of *how mathematics (often) advances*.

Our understanding of objects in a reasonably-uncomplicated realm forms the basis of relational facts, and then those relational facts form the basis of our understanding of object in a slightly-more-complicated realm; the tail, as they say, wags the dog. And this pattern is repeated throughout mathematics; witness the advent of complex numbers, fractional (even negative) dimensions and factorials, Lebesgue integration, and ... the trigonometric functions.

One reason to explore Quadrant I thoroughly before moving on is that almost (if not absolutely) every worthwhile identity has an elegant “picture-proof” in that context. The bulk of this paper (hopefully) shows that the Complete Triangle serves as the picture for many such proofs. Others abound. For example, this article

<http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html>

gives some picture-proofs (though not my favorite one) of the Angle-Addition Formulas⁸

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \qquad \cos(A + B) = \cos A \cos B - \sin A \sin B$$

I use these formulas as the springboard for expanding our angular horizons.

If we were at all squeamish about our “degenerate” configurations in the Complete Triangle, for instance, we might reasonably view our presumed values of $\sin 90^\circ$ and $\cos 90^\circ$ with some suspicion. However, by assigning $A = 30^\circ$ and $B = 60^\circ$, the above formulas *confirm* those suspect values using *irrefutable* trig values for non-degenerate angles A and B ; that is, while the left-hand sides of the formulas might be “iffy”, the right-hand sides are perfectly well-defined for the given A and B , and the resulting computation should remove any lingering doubt about the true nature of $\sin 90^\circ$ and $\cos 90^\circ$. (Likewise, the Angle-Subtraction Formulas confirm the values of $\sin 0^\circ$ and $\cos 0^\circ$.)

More dramatically, in a context where $\sin \theta$ and $\cos \theta$ are lengths of segments in a right triangle having an angle θ , the expressions “ $\sin 120^\circ$ ” and “ $\cos 120^\circ$ ” are laughably non-sensical. But the Angle-Addition Formula tells us that, *if “ $\cos 120^\circ$ ” is to equal ANYTHING, then it must be equal*

⁸Drifting a bit off topic, here’s a “cheer” to help remember the Angle-Addition and -Subtraction Formulas:

Sine! Cosine! Sign! Cosine! Sine!
Cosine, cosine, co-sign, sine, sine!

Sprinkling in A s and B s as necessary, the lines encode the right-hand sides of the formulas for $\sin(A \pm B)$ and $\cos(A \pm B)$, where we take “sign” to indicate the *same sign* (plus or minus) as in the combined argument from the left-hand side, and “co-sign” indicates the *opposite sign* (minus or plus):

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B & \text{sign : } \pm \rightarrow \pm \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B & \text{co - sign : } \pm \rightarrow \mp \end{aligned}$$

to $-\frac{1}{2}$ (the result of entering $A = B = 60^\circ$ into the right-hand side of the formula); likewise the expression “ $\sin 120^\circ$ ” can be given meaning. More generally, however, having come to grips with the trig values at 90° , we can extend our understanding throughout all of Quadrant II via these computations:

$$\begin{aligned}\sin(A + 90^\circ) &= \sin A \cos 90^\circ + \cos A \sin 90^\circ = \cos A \\ \cos(A + 90^\circ) &= \cos A \cos 90^\circ - \sin A \sin 90^\circ = -\sin A\end{aligned}$$

whose right-hand sides are, once more, are perfectly well-defined for acute (even right) A . And with an understanding of Quadrants I and II, we can spill into Quadrants III and IV (and beyond), arriving in the end with the traditional sign assignments and the any-angle structure of the Complete Triangle.

Of course, this Quadrant-by-Quadrant approach is inefficient, and it certainly isn't worth spending an *inordinate* amount of time on. (Some of the concepts can make for thought-provoking homework exercises.) After establishing the gist of the process, liberal hand-waving is more than appropriate. The point is to take the opportunity (if only in passing) to present some sense of the journey⁹ in the development of Trig, showing it—and Mathematics in general—not as a collection of pre-fabricated definitions but an ever-lengthening chain of “what if” questions and their answers.

⁹While certainly outside the scope of a course that introduces Trigonometry to students, it's worth pointing out here that the journey continues: the Power Series Properties not only form the bridge to the realm in which “ $\sin i$ ” and “ $\cos i$ ” have meaning, they also pave the way for Euler's Formula and the polar representation for complex numbers and all the mathematics that comes from that.