AN EXTENSION OF A THEOREM OF BARLOTTI TO MULTIPLE DIMENSIONS

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In [1], A. Barlotti shows that any affinely regular polygon in the plane can be expressed as the "vertex sum" of two regular polygons in that plane. (See Figure 1.) This note recasts Barlotti's result as a straightforward matrix decomposition and broadens the geometric application to arbitrary point-multisets in any dimension.

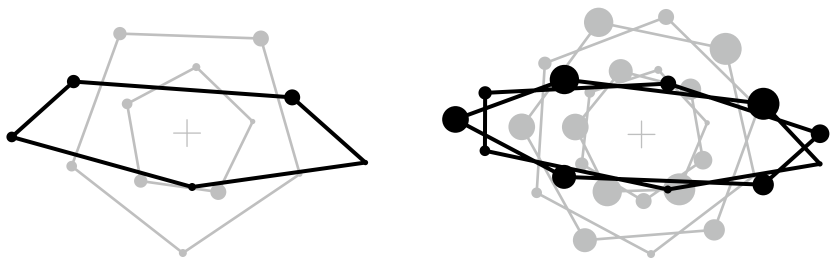

FIGURE 1. Affinely regular polygons (black) and their regular "Barlotti components" (gray). The position vector of each black vertex is the sum of vectors for gray vertices of corresponding size.

We note in passing a related result. (See, for instance, [3].)

Theorem. *(The Napoleon-Barlotti Theorem) Given an affinely regular planar polygon, construct regular polygons of the same type on all sides of the given polygon, either all "outside" or all "inside". The centroids of the constructed polygons lie at the vertices a new regular polygon of the given type; moreover, the "inside" and "outside" results are traces in opposing orientations.*

When applied to triangles, the above is **Napoleon's Theorem**, and the resulting figures are known as the "Napoleon triangles" of the original.

As regular figures, a polygon's Barlotti components and the polygons arising from the Napoleon-Barlotti constructions are clearly similar. Interestingly, the exact transformations that carry members of one pair onto members of another pair involve a rotation and a dilation, whose parameters of these depend only upon the polygons' number of edges.

The Napoleon-Barlotti Theorem, therefore, effectively provides a geometric construction for the Barlotti components of a planar polygon. The notion is intriguing, but its generalization to higher-dimensional figures is difficult to conceptualize. After all, the two components of a polygon's decomposition arise from two naturally complementary constructions, one "inside" and one "outside". As we show (and one might expect), a tetrahedron's Barlotti decomposition has three regular

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components. Thus, even if one could find the appropriate analogues of the Napoleon constructions, what would be their natural application to tetrahedra? An obvious "inside-outside" dichotomy would leave one needed component un-accounted-for. Moreover, Barlotti decompositions, while unique in the plane, are numerous in dimensions 3 and above; would this mean that a generalized Napoleon construction would admit variants, one for each possible decomposition, or would it single out a particularly "natural" decomposition? We have no answers to these questions.

The point of this digression is that, although we can very easily extend Barlotti's decomposition theorem, we may have lost the satisfying geometric connection that Napoleon's theorem offers.

1. Vertex Sums Defined

Given two n-gons A and B with consecutive vertices (represented by dots of increasing size in the figures) $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, respectively, their *vertex sum* $C := A \oplus B$ is the polygon with vertices implied by the relation $[c_i] = [a_i] + [b_i]$, where $[p]$ represents the co-ordinate (column) m-vector identified with the point p in $\mathbb{R}^m$.

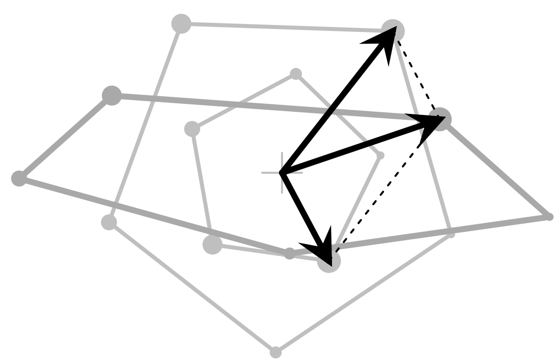

Figure 2. The mechanism for computing a vertex sum.

Writing $[P]$ for the matrix whose i-th column is the co-ordinate vector of the i-th vertex of n-gon P, we have the vertex sum succinctly defined as a matrix sum: $[A \oplus B] = [A] + [B]$.

2. Barlotti's Theorem

Barlotti's Theorem concerns regular polygons and affinely regular polygons in the plane. Our analysis requires that we "matrixify" these definitions. Note that, throughout this section, all polygons mentioned are assumed to live in $\mathbb{R}^2$, and all coordinate matrices are assumed to have two rows.

Define P_n, the *standard* regular planar n-gon[1], to be such that the i-th column of $[P_n]$ has the form

$$[\cos(2i\pi/n), \sin(2i\pi/n)]^\top$$

[1]Barlotti's Theorem in fact applies to regular "$\{n/k\}$-gons" ($k = 0, 1, \ldots, n-1$) of all types: convex ($k = 1$ or $n-1$), starry (k relatively prime to n), multiply-traced (k a factor of n), and even collapsed-to-a-single-point ($k = 0$). (The right-hand side of Figure 1 depicts a $\{11/2\}$-gon.) For these, the i-th column of the coordinate matrix of the *standard* regular planar $\{n/k\}$-gon uses the angle $2ik\pi/n$.

An arbitrary regular planar n-gon can be obtained from P_n via the application of an orthogonal transformation (rotation and/or reflection), and then a dilation, and then a translation. In matrix terms,

Definition 1. *P is a regular planar n-gon if and only if*

$$[P] = q\ \mathbf{Q}[P_n] + \mathbf{t}\ \mathbf{1}_n$$

where $\mathbf{Q}$ is a real orthogonal 2×2 matrix (effecting the orthogonal tranformation), q is a real scalar (effecting the dilation), $\mathbf{t}$ is a real (translation) vector, and $\mathbf{1}_n$ is the 1-by-n "all 1s" matrix.

An *affinely regular n-gon* is the image of P_n under an affine transformation; thus,

Definition 2. *P is an affinely regular planar n-gon if and only if*

$$[P] = \mathbf{M}[P_n] + \mathbf{t}\ \mathbf{1}_n$$

where $\mathbf{M}$ is a real 2×2 matrix and $\mathbf{t}$ is a real (translation) vector.

With these definitions made, we can re-state —and matrixify— Barlotti's Theorem, and prove the geometric result from the algebraic one.

Theorem 1. *(Barlotti's Theorem) An affinely regular planar n-gon can be expressed as the vertex sum of two regular planar n-gons, called the* Barlotti components *of the original polygon.*

Theorem 1-M. *(Barlotti's Theorem, matrix form) Given a real, 2×2 matrix $\mathbf{M}$, there exist real, 2×2 orthogonal matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$, and real scalars q_1 and q_2, such that*

$$\mathbf{M} \ = \ q_1\mathbf{Q}_1 \ + \ q_2\mathbf{Q}_2$$

Proof of Theorem 1-M. The Singular Value Decomposition Theorem allows us to express $\mathbf{M}$ as $\mathbf{USV}^\top$, where $\mathbf{U}$ and $\mathbf{V}^\top$ are orthogonal matrices and $\mathbf{S} = diag(s_1, s_2)$ is a diagonal matrix (of $\mathbf{M}$'s singular values). Now, make the following assignments:

$$\begin{aligned} &\mathbf{Q}'_1 := diag(1,1) && \mathbf{Q}_1 := \mathbf{U}\mathbf{Q}'_1\mathbf{V}^\top && q_1 := (\ \ s_1 + s_2)/2 \\ &\mathbf{Q}'_2 := diag(-1,1) && \mathbf{Q}_2 := \mathbf{U}\mathbf{Q}'_2\mathbf{V}^\top && q_2 := (-s_1 + s_2)/2 \end{aligned}$$

Then $q_1\mathbf{Q}_1 + q_2\mathbf{Q}_2 \ = \ \mathbf{U}\,(q_1\mathbf{Q}'_1 + q_2\mathbf{Q}'_2)\,\mathbf{V}^\top \ = \ \mathbf{U}(\mathbf{S})\mathbf{V}^\top \ = \ \mathbf{M}$. As $\mathbf{Q}'_1$ and $\mathbf{Q}'_2$ are orthogonal matrices, and as the product of orthogonal matrices the orthogonal, we find that $\mathbf{Q}_1$ and $\mathbf{Q}_2$, as defined, satisfy the requirements of the theorem. □

Proof of Theorem 1. Given P affinely regular, we have

$$\begin{aligned} [P] &= \mathbf{M}[P_n] + \mathbf{v}\ \mathbf{1}_n \\ &= (q_1\mathbf{Q}_1 + q_2\mathbf{Q}_2)\,[P_n] + (\mathbf{v} + \mathbf{0})\,\mathbf{1}_n \\ &= (q_1\mathbf{Q}_1[P_n] + \mathbf{v}\ \mathbf{1}_n) + (q_2\mathbf{Q}_2[P_n] + \mathbf{0}\mathbf{1}_n) \\ &=: [Q_1] + [Q_2] \end{aligned}$$

Therefore, affinely regular polygon P is the vertex sum of regular polygons Q_1 and Q_2. □

Note: One can show that the decomposition $\mathbf{M} = q_1\mathbf{Q}_1 + q_2\mathbf{Q}_2$ is unique up to the order of the summands. If we suppress the translational concerns (say, by assuming all figures are centered[2] at the origin), then we see that the *sizes* and *orientations* of the regular polygons in the Barlotti decomposition are uniquely determined.

3. Barlotti's Theorem in Multiple Dimensions

As the arguments of previous section suggest, Barlotti's Theorem actually speaks to a decomposition of *matrices* (and/or the geometric transformations corresponding to them); the involvement of polygons, though geometrically appealing, turns out to be incidental. The matrix result is readily extensible to arbitrarily many dimensions:

Theorem 2-M. *(Barlotti's Theorem in Multiple Dimensions, matrix form) A $d \times d$ matrix can be expressed as the sum of d or fewer[3] "scaled orthogonal" $d \times d$ matrices:*

$$\mathbf{M} = q_1\,\mathbf{Q}_1 + q_2\,\mathbf{Q}_2 + \cdots + q_d\,\mathbf{Q}_d$$

Proof. As before, we observe that $\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^\top$, with orthogonal $\mathbf{U}$ and $\mathbf{V}^\top$, and with diagonal $\mathbf{S} = diag(s_1, s_2, \ldots, s_d)$. We assign (for example)

$$\begin{array}{lrrrcrrl}
\mathbf{Q}'_1 := diag(& 1, & 1, & 1, & \ldots, & 1 & 1 &), \quad \mathbf{Q}_1 := \mathbf{U}\mathbf{Q}'_1\mathbf{V}^\top, \quad q_1 := (\ s_1 + s_d\ )/2 \\
\mathbf{Q}'_2 := diag(& -1, & 1, & 1, & \ldots, & 1 & 1 &), \quad \mathbf{Q}_2 := \mathbf{U}\mathbf{Q}'_2\mathbf{V}^\top, \quad q_2 := (\ s_2 - s_1\ )/2 \\
\mathbf{Q}'_3 := diag(& -1, & -1, & 1, & \ldots, & 1 & 1 &), \quad \mathbf{Q}_3 := \mathbf{U}\mathbf{Q}'_3\mathbf{V}^\top, \quad q_3 := (\ s_3 - s_2\ )/2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \qquad\qquad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \\
\mathbf{Q}'_d := diag(& -1, & -1, & -1, & \ldots, & -1 & 1 &), \quad \mathbf{Q}_d := \mathbf{U}\mathbf{Q}'_d\mathbf{V}^\top, \quad q_d := (\ s_d - s_{d-1}\ )/2
\end{array}$$

Then $q_1\mathbf{Q}_1 + q_2\,\mathbf{Q}_2 + \cdots + q_d\,\mathbf{Q}_d = \mathbf{U}\,(q_1\mathbf{Q}'_1 + q_2\,\mathbf{Q}'_2 + \cdots + q_d\,\mathbf{Q}'_d)\,\mathbf{V}^\top = \mathbf{U}(\mathbf{S})\mathbf{V}^\top = \mathbf{M}$, giving the proposed decomposition. □

Note: When the number of summands is greater than 2, the decomposition is not unique.

The application of this matrix theorem to vertex sums of geometric multiple-dimensional figures follows just as in the two-dimensional case, and we have statements such as

- Any tetrahedron (or d-simplex) is the vertex sum of three (d) regular tetrahedra (d-simplices).
- Any four-dimensional (d-dimensional) parallelotope is the vertex sum of four (d) hypercubes (d-cubes).

[2] The *center* of a polygon is the point corresponding to the average of the coordinate vectors of the polygon's vertices.

[3] If the rank, r, of the matrix is less than d, we can reduce the maximum number of summands required to $r+1$: If $r = 0$, then $\mathbf{M}$ is the zero matrix and we can assign $q_1 = 0$ and take $\mathbf{Q}_1$ to be any orthogonal matrix (say, the identity matrix), and the reduced bound ($r + 1 = 1$) on the number of summands holds. If $r > 0$, then $d - r$ of the singular values (say, s_1 to s_{d-r}) of $\mathbf{M}$ are zero; in the argument given in the proof, the $d - r - 1$ values q_2 through q_{d-r} also vanish, leaving at most $r + 1$ non-zero values and thus $r + 1$ terms in the matrix decomposition. That we cannot, in general, do better than this is evident from the example $\mathbf{M} := diag(1, 0)$, which has $d = 2$ and $r = 1$; the matrix is clearly not scaled-orthogonal and therefore cannot be expressed as the sum of fewer than $r + 1 = 2$ scaled-orthogonal matrices.

These statements, however, ignore the full power of the matrix result. To take proper advantage, we must overcome the short-sightedness of vertex sums as defined in Section 1.

3.1. **From Vertex Sums to Point Sums.** The vertex sum of polygons A and B, although defined specifically on the vertices of those polygons, in fact extends linearly to all points along the polygonal paths that make up A and B.

For instance, the midpoint M_C of segment c_1c_2 in polygon $C := A \oplus B$ satisfies $[M_C] = [M_A] + [M_B]$, where M_A and M_B are midpoints of corresponding segments in polygons A and B. The same type of statement is true for any point along the polygonal path of C.

Indeed, the vertex sum may be extended across the entire *linear span* of the polygons' vertices[4] via this defining relation

$$k_1[c_1] + k_2[c_2] + \cdots + k_n[c_n] = k_1([a_1] + [b_1]) + k_2([a_2] + [b_2]) + \cdots + k_n([a_n] + [b_n])$$

In this context (and assuming for the moment a non-degenerate case), we envision each polygon not as a set of vertices, not as a continuous path, and not even as a bounded planar region, but as full copy of $\mathbb{R}^2$ "watermarked" with the polygonal path whose vertices serve to coordinatize that plane; the extended vertex sum imposes a coordinatization of the resultant polygon's copy of $\mathbb{R}^2$.

What we arrive at, then, is not a mere "vertex sum" defined on polygons but a general "point sum" defined across the whole plane, or, by extension, the whole of any d-dimensional space. Perhaps most generally, this "point sum" applies to arbitrary multi-sets of points we can take from any $\mathbb{R}^d$. We can state the multi-dimensional Barlotti's Theorem in terms of such point-sums.

Theorem 2. *(Barlotti's Theorem in Multiple Dimensions) The affine image of a point multi-set of d dimensions may be expressed as the point sum of d similar images of the original point multi-set. That is, given a point multi-set P_0 with elements spanning $\mathbb{R}^d$, suppose $P = T(P_0) := \{T(p) \mid p \in P_0\}$, where T is an affine transformation, then*

$$P = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_d$$

where each "Barlotti Component" is defined by $Q_i = S_i(P_0) := \{S_i(p) \mid p \in P_0\}$ for S_i some similarity transformation (an isometry, plus dilation, plus translation).

Theorem 2 follows from Theorem 2-M in the same way that Theorem 1 follows from Theorem 1-M, save that the matrix $[P_n]$ from that proof should be replaced by a matrix of whose columns span the set P.

Note: The number of "Barlotti components" depends upon the dimension of the linear span of the point multi-set, even if that multi-set is embedded in a larger-dimensional space.

The Barlotti Theorem in Multiple Dimensions has a consequence related to the Petr-Douglas-Neumann Theorem (see [2]), which decomposes *any* polygon —not necessarily affinely regular, and not necessarily even planar— into a vertex sum of regular polygons. This result will be taken up in a separate note.

[4]Of course, with our planar setting, at most two vertices from each polygon suffice to form a spanning set, so we needn't involve the entire collection of the formula given.

4. Examples

Tetrahedra and Parallelopipeds. *Any tetrahedron (being the affine image of a regular tetrahedron) is the point-sum of three regular tetrahedra. Any parallelopiped (being the affine image of a cube) is the point-sum of three cubes.* Similarly for higher-dimensional analogues of these figures. Note that the result is true whether we think of the figures in question as solids, hollow shells, skeletal arrangements of edges, collections of vertices, or "watermarked" copies of $\mathbb{R}^3$.

Backward (Classical) Barlotti. (Figure 3.) *A regular planar n-gon is the point sum of two similar copies of any non-degenerate affine image of the n-gon.*

Proof: If affinely regular polygon Q is a non-degenerate affine image of regular polygon P, then Q is an affine image of P. (The non-degenerateness condition on Q is equivalent to the condition that the affine transformation from P to Q can be inverted.) The (forward) Barlotti Theorem then applies.

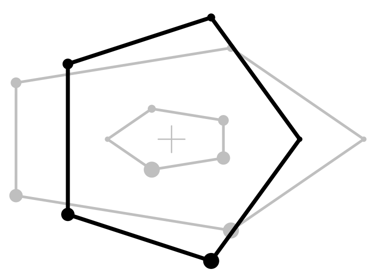

Figure 3. Regular pentagon as point-sum of two affinely-regular pentagons.

Circular Barlotti, Forward and Backward. (Figures 4 and 5.) *An ellipse is a point-sum of two circles; and a circle is the point-sum of two similar copies of any non-degenerate ellipse.*

Proof: An ellipse is the affine image of a circle, and any two circles are similar.

More generally, a d-dimensional ellipsoid is the point-sum of d d-dimensional spheres, and any d-dimensional sphere is the point-sum of d similar non-degenerate d-dimensional ellipses.

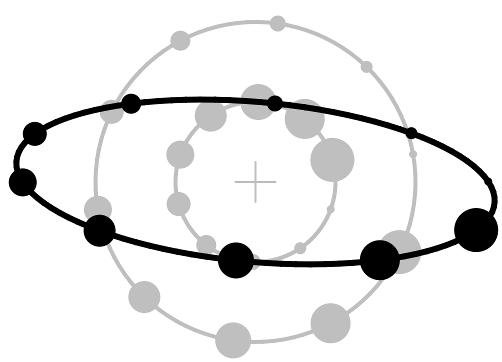

Figure 4. Ellipse as point-sum of two circles.

Simplex Decomposition. *Any (multi-)set of d points is the point-sum of vertices from $(d-1)$ regular $(d-1)$-dimensional simplices.*

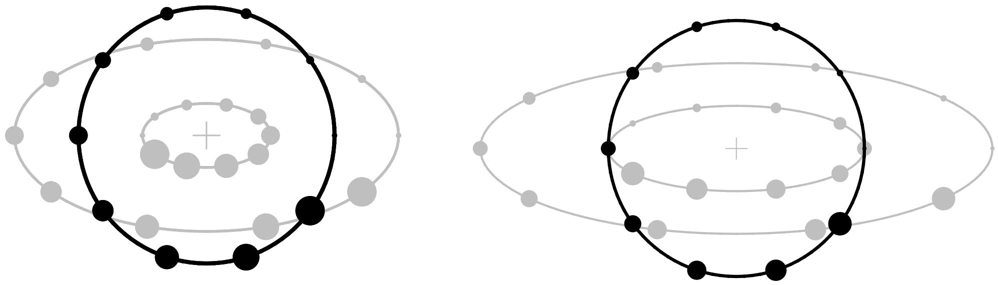

Figure 5. Circles as point-sums of ellipses.

This follows from the fact that any set of d points —which we may embed in $\mathbb{R}^{d-1}$— is the affine image of the set of vertices of a $(d-1)$-dimensional simplex. To prove the fact, temporarily add a gratuitous dimension so that $[P]$, the coordinate matrix of our set, is a d-by-d matrix. Then, obviously, we have the matrix relation $[P] = [P]\ \mathbf{I}_d$, where $\mathbf{I}_d$ is the d-by-d identity matrix. But, $\mathbf{I}_d$ is *also* a coordinate matrix of a $(d-1)$-dimensional simplex in $\mathbb{R}^d$. Interpreting the $[P]$ on the right as a transformation matrix acting on the simplex, we see that our point set is the affine (in fact, *linear*) image of the vertices of the simplex. With this geometric relationship established, we are guaranteed an affinity from the simplex to our point set even after both have been re-embedded in $\mathbb{R}^{d-1}$.

References

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