

# HEDRONOMETRIC FORMULAS FOR A HYPERBOLIC TETRAHEDRON

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ABSTRACT. This “living document” will serve as an ever-expanding resource of results in hyperbolic (tetra)hedronometry. To maximize information density, I minimize discussion and proof (at least in early drafts), but the reader can readily verify the geometry with basic plane trigonometry and the algebra with standard identities; the only truly sophisticated notions are the Schlafli and Derevnin-Mednykh formulas for volume.

*Refer to Appendix A, “Standard Notation”, for explanation of symbols used throughout, and to Figure 1 for the presumed arrangement of tetrahedral elements.*

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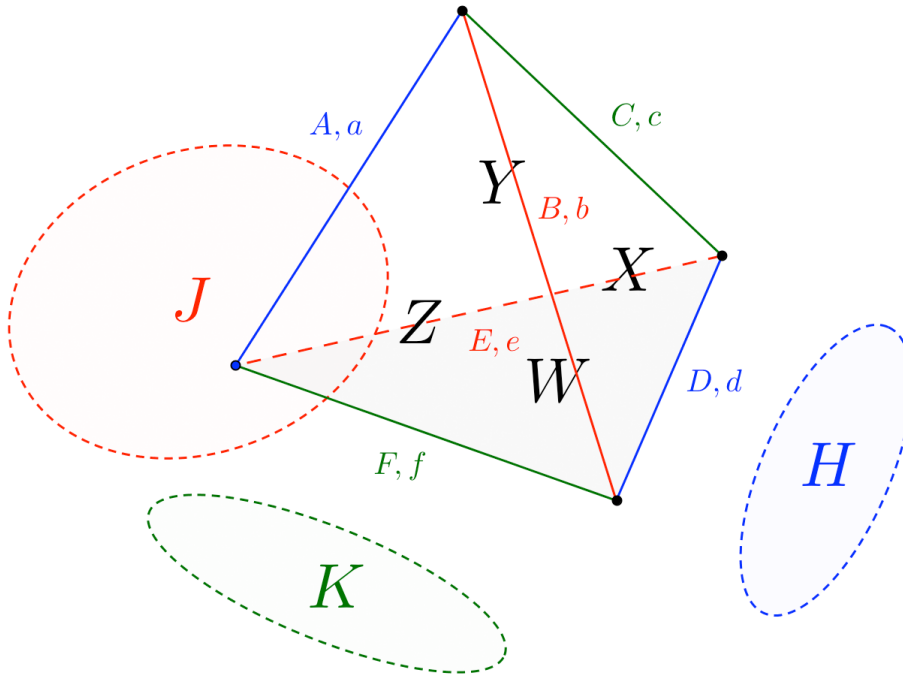


FIGURE 1. Tetrahedral Elements: Faces ( $W, X, Y, Z$ ), dihedral angles ( $A, B, C, D, E, F$ ), pseudofaces ( $H, J, K$ ), and edges ( $a, b, c, d, e, f$ ).

### 1. THE LAWS OF COSINES

At the core of hyperbolic hedronometry lie three Laws of Cosines that epitomize the field as “dimensionally-enhanced” trigonometry; they relate, not the edges and planar angles of a triangle, but the *faces* and *dihedral angles* of a *tetrahedron*. The first of these laws, in particular, also formally defines the figure’s geometrically-amorphous<sup>1</sup> but algebraically-rewarding<sup>2</sup> *pseudofaces* associated with pairs of opposite edges.

<sup>1</sup>In Euclidean space, a pseudoface is a quadrilateral determined by projection of the tetrahedron into a plane parallel to a pair of opposite edges; no geometric interpretation of a general hyperbolic pseudoface is known. (See Section 7 for candidates in some special cases.) Nevertheless, here we simply *declare* a pseudoface to be an element whose “area” satisfies (1a), and we *declare* that its “area” is bounded as if the element were a hyperbolic quadrilateral.

<sup>2</sup>I doubt that the significance of pseudofaces in catalyzing tetrahedral analysis can be overstated.

**Theorem 1.1** (Law of Cosines for Opposite Dihedral Angles; “Law of Opposite Cosines”).

$$(1a) \quad \begin{aligned} \ddot{Y}_2\ddot{Z}_2 + \overline{Y_2Z_2}\ddot{A} &= \ddot{H}_2 = \ddot{W}_2\ddot{X}_2 + \overline{W_2X_2}\ddot{D} \\ \ddot{Z}_2\ddot{X}_2 + \overline{Z_2X_2}\ddot{B} &= \ddot{J}_2 = \ddot{W}_2\ddot{Y}_2 + \overline{W_2Y_2}\ddot{E} \\ \ddot{X}_2\ddot{Y}_2 + \overline{X_2Y_2}\ddot{C} &= \ddot{K}_2 = \ddot{W}_2\ddot{Z}_2 + \overline{W_2Z_2}\ddot{F} \end{aligned}$$

for values  $H, J, K$  that we declare to lie between 0 and  $2\pi$  (inclusive).

**Theorem 1.2** (Law of Cosines for All Dihedral Angles; “Law of All Cosines”).

$$(1b) \quad \begin{aligned} 0 &= 1 - \ddot{W}_2^2 - \ddot{X}_2^2 - \ddot{Y}_2^2 - \ddot{Z}_2^2 - 4\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2 - \ddot{H}_2^2 - \ddot{J}_2^2 - \ddot{K}_2^2 - 2\ddot{H}_2\ddot{J}_2\ddot{K}_2 \\ &+ 2\ddot{H}_2(\ddot{W}_2\ddot{X}_2 + \ddot{Y}_2\ddot{Z}_2) + 2\ddot{J}_2(\ddot{W}_2\ddot{Y}_2 + \ddot{Z}_2\ddot{X}_2) + 2\ddot{K}_2(\ddot{W}_2\ddot{Z}_2 + \ddot{X}_2\ddot{Y}_2) \end{aligned}$$

**Theorem 1.3** (Law of Cosines for Adjacent Dihedral Angles; “Law of Adjacent Cosines”).

$$(1c) \quad \ddot{W}_2 = \ddot{X}_2\ddot{Y}_2\ddot{Z}_2 - \overline{X_2Y_2Z_2} \sqrt{-4\llbracket A, B, C \rrbracket} + \overline{X_2Y_2Z_2}\ddot{A} + \overline{X_2Y_2Z_2}\ddot{B} + \overline{X_2Y_2Z_2}\ddot{C}$$

When the edges opposite a face, say  $W$ , are mutually orthogonal (equivalently, when  $X, Y, Z$  are right triangles with hypotenuses bounding  $W$ , and when  $A = B = C = \pi/2$ ), the last of these reduces nicely:

**Corollary 1.4** (The Pythagorean Theorem for Right-Corner Tetrahedra).

$$(2) \quad \ddot{W}_2 = \ddot{X}_2\ddot{Y}_2\ddot{Z}_2 - \overline{X_2Y_2Z_2}$$

The Law of Opposite Cosines is especially-significant, as it allows us to express the measures of the dihedral angles of our tetrahedron in terms of the areas of its faces and pseudofaces. Given the role dihedral angles play in governing the shape of a tetrahedron, we arrive at the fundamental motivation for the study of hedronometry:<sup>3</sup>

*A hyperbolic tetrahedron is uniquely determined (up to isometry)  
by its face and pseudoface areas.*

We can, therefore, express *all* metrics of our tetrahedron —edge lengths, volume, Gram invariants, and so forth— in terms of these areas, with varying degrees of succinctness. The purpose of this note is to document such expressions.

<sup>3</sup>One observes that the independence of the dihedral angles contribute *six* degrees of freedom to the shape of a tetrahedron. While there are *seven* faces and pseudofaces, the Law of All Cosines provides a dependence reducing the degrees of freedom to six.

2.  $\llbracket H_2, W_2, X_2 \rrbracket$  AND  $\langle W_2 \rangle$  AND  $\langle H_2 \rangle$ 

From (1a) comes this Heronic sine product relation:

$$(3a) \quad \llbracket H_2, W_2, X_2 \rrbracket = \frac{1}{4} \overline{W_2^2 X_2^2 D^2} \geq 0$$

More atomically:

$$(3b) \quad \overline{W_2 X_2} \ddot{D}_2^2 = \overline{(H_4 + W_4 + X_4)} \overline{(-H_4 + W_4 + X_4)}$$

$$(3c) \quad \overline{W_2 X_2} \overline{D_2^2} = \overline{(H_4 - W_4 + X_4)} \overline{(H_4 + W_4 - X_4)}$$

We introduce some handy symbolic abbreviations<sup>4</sup> for key hedronometric expressions:

$$(4a) \quad \langle W_2 \rangle := -\ddot{W}_2 - 2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2 + \ddot{H}_2\ddot{X}_2 + \ddot{J}_2\ddot{Y}_2 + \ddot{K}_2\ddot{Z}_2 \geq 0$$

$$(4b) \quad \langle H_2 \rangle := \ddot{H}_2 + \ddot{J}_2\ddot{K}_2 - \ddot{W}_2\ddot{X}_2 - \ddot{Y}_2\ddot{Z}_2 \geq 0$$

with  $\langle X_2 \rangle$ ,  $\langle Y_2 \rangle$ ,  $\langle Z_2 \rangle$ ,  $\langle J_2 \rangle$ ,  $\langle K_2 \rangle$  defined similarly. Reduction *modulo* the Law of All Cosines (1b) affords us these formulas:

$$(5a) \quad 4 \langle H_2 \rangle \llbracket H_2, W_2, X_2 \rrbracket = \langle J_2 \rangle \langle K_2 \rangle - \langle Y_2 \rangle \langle Z_2 \rangle$$

$$(5b) \quad 4 \langle H_2 \rangle \llbracket H_2, Y_2, Z_2 \rrbracket = \langle J_2 \rangle \langle K_2 \rangle - \langle W_2 \rangle \langle X_2 \rangle$$

and these Law-of-Cosines-like relations:

$$(6a) \quad \langle H_2 \rangle^2 + \langle X_2 \rangle^2 - 2\ddot{W}_2 \langle H_2 \rangle \langle X_2 \rangle = 16 \llbracket J_2, W_2, Y_2 \rrbracket \llbracket K_2, W_2, Z_2 \rrbracket$$

$$(6b) \quad \langle H_2 \rangle^2 + \langle W_2 \rangle^2 - 2\ddot{X}_2 \langle H_2 \rangle \langle W_2 \rangle = 16 \llbracket J_2, Z_2, X_2 \rrbracket \llbracket K_2, X_2, Y_2 \rrbracket$$

$$(6c) \quad \langle J_2 \rangle^2 + \langle K_2 \rangle^2 - 2\ddot{H}_2 \langle J_2 \rangle \langle K_2 \rangle = 16 \llbracket H_2, W_2, X_2 \rrbracket \llbracket H_2, Y_2, Z_2 \rrbracket$$

whence

$$(7a) \quad \ddot{W}_2 = \frac{\langle X_2 \rangle \langle J_2 \rangle \langle K_2 \rangle + \langle H_2 \rangle \langle Y_2 \rangle \langle K_2 \rangle + \langle H_2 \rangle \langle J_2 \rangle \langle Z_2 \rangle - \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle}{2 \langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle}$$

$$(7b) \quad \ddot{X}_2 = \frac{\langle W_2 \rangle \langle J_2 \rangle \langle K_2 \rangle + \langle H_2 \rangle \langle Z_2 \rangle \langle K_2 \rangle + \langle H_2 \rangle \langle J_2 \rangle \langle Y_2 \rangle - \langle W_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle}{2 \langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle}$$

$$(7c) \quad \ddot{H}_2 = \frac{\langle H_2 \rangle^2 \langle J_2 \rangle^2 + \langle H_2 \rangle^2 \langle K_2 \rangle^2 - (\langle J_2 \rangle \langle K_2 \rangle - \langle W_2 \rangle \langle X_2 \rangle)(\langle J_2 \rangle \langle K_2 \rangle - \langle Y_2 \rangle \langle Z_2 \rangle)}{2 \langle H_2 \rangle^2 \langle J_2 \rangle \langle K_2 \rangle}$$

As a result, we can conceivably express tetrahedral metrics entirely in terms of  $\langle W_2 \rangle$ ,  $\langle X_2 \rangle$ ,  $\langle Y_2 \rangle$ ,  $\langle Z_2 \rangle$ ,  $\langle H_2 \rangle$ ,  $\langle J_2 \rangle$ ,  $\langle K_2 \rangle$ . Equations (7) show that “entirely” may be somewhat cumbersome, so some direct reference to sines and cosines of (half-)faces appear often in our formulas, but we’ll see that these abbreviations tend to dominate.

<sup>4</sup>Non-negativity follows from (10).

## 3. FROM LENGTHS TO AREAS, AND BACK AGAIN

Standard formulas give the area of a tetrahedron's face from lengths of bounding edges

$$(8) \quad \ddot{W}_2 = \frac{\ddot{d}_2^2 + \ddot{e}_2^2 + \ddot{f}_2^2 - 1}{2\ddot{d}_2\ddot{e}_2\ddot{f}_2} \quad \overline{W}_2 = \frac{\sqrt{[d, e, f]}}{2\ddot{d}_2\ddot{e}_2\ddot{f}_2}$$

and we can determine pseudoface areas from lengths of edges in conspicuous pairs

$$(9a) \quad \ddot{H}_2 = \frac{-\overline{a}_2^2\overline{d}_2^2 + \overline{b}_2^2\overline{e}_2^2 + \overline{c}_2^2\overline{f}_2^2}{2\overline{b}_2\overline{e}_2\overline{c}_2\overline{f}_2} \quad \overline{H}_2 = \frac{\sqrt{[2\overline{a}_2\overline{d}_2, 2\overline{b}_2\overline{e}_2, 2\overline{c}_2\overline{f}_2]}}{2\overline{b}_2\overline{e}_2\overline{c}_2\overline{f}_2}$$

$$(9b) \quad \ddot{H}_4^2 = \frac{-\overline{a}_2^2\overline{d}_2^2 + (\overline{b}_2\overline{e}_2 + \overline{c}_2\overline{f}_2)^2}{4\overline{b}_2\overline{e}_2\overline{c}_2\overline{f}_2} \quad \overline{H}_4^2 = \frac{\overline{a}_2^2\overline{d}_2^2 - (\overline{b}_2\overline{e}_2 - \overline{c}_2\overline{f}_2)^2}{4\overline{b}_2\overline{e}_2\overline{c}_2\overline{f}_2}$$

From these, we can verify

$$(10) \quad \begin{aligned} \langle W_2 \rangle \ddot{a}_2 \ddot{b}_2 \ddot{c}_2 &= \langle X_2 \rangle \ddot{a}_2 \ddot{e}_2 \ddot{f}_2 = \langle Y_2 \rangle \ddot{d}_2 \ddot{b}_2 \ddot{f}_2 = \langle Z_2 \rangle \ddot{d}_2 \ddot{e}_2 \ddot{c}_2 \\ &= \langle H_2 \rangle \ddot{a}_2 \ddot{d}_2 = \langle J_2 \rangle \ddot{b}_2 \ddot{e}_2 = \langle K_2 \rangle \ddot{c}_2 \ddot{f}_2 = \frac{-\widehat{g}}{64\overline{a}_2\overline{b}_2\overline{c}_2\overline{d}_2\overline{e}_2\overline{f}_2} \end{aligned}$$

Consequently, we have hedronometric formulas<sup>5</sup> for edge lengths:

$$(11a) \quad \ddot{a}_2^2 = \frac{\langle J_2 \rangle \langle K_2 \rangle}{\langle W_2 \rangle \langle X_2 \rangle} \quad \overline{a}_2^2 = \frac{4\langle H_2 \rangle [H_2 Y_2 Z_2]}{\langle W_2 \rangle \langle X_2 \rangle} \quad \widehat{a}_2^2 = \frac{4\langle H_2 \rangle [H_2 Y_2 Z_2]}{\langle J_2 \rangle \langle K_2 \rangle}$$

$$(11b) \quad \ddot{d}_2^2 = \frac{\langle J_2 \rangle \langle K_2 \rangle}{\langle Y_2 \rangle \langle Z_2 \rangle} \quad \overline{d}_2^2 = \frac{4\langle H_2 \rangle [H_2 W_2 X_2]}{\langle Y_2 \rangle \langle Z_2 \rangle} \quad \widehat{d}_2^2 = \frac{4\langle H_2 \rangle [H_2 W_2 X_2]}{\langle J_2 \rangle \langle K_2 \rangle}$$

and, in turn, a symmetric equality involving opposing pairs of edges<sup>6</sup>

$$(12) \quad \frac{\overline{a}^2 \overline{d}^2}{[H_2 Y_2 Z_2][H_2 W_2 X_2]} = \frac{\overline{b}^2 \overline{e}^2}{[J_2 Z_2 X_2][J_2 W_2 Y_2]} = \frac{\overline{c}^2 \overline{f}^2}{[K_2 X_2 Y_2][K_2 W_2 Z_2]} \\ = \left( \frac{16\langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle}{\langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle} \right)^2$$

<sup>5</sup>The sine formulas derive from the cosine formulas with the help of (5).

<sup>6</sup>A cleaner version of this result appears in (16b).

## 4. GRAM RELATIONS

Consider the face-indexed (“angle”) Gram matrix,  $G$ , and conjugate (“edge”) Gram matrix,  $g$ , detailed in Appendix A.3. Their determinants —and the *conjugation ratio*  $\Gamma$ — have the following hedronometric forms:

$$(13a) \quad \widehat{G} := \det G = \frac{-4\langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle}{\overline{W_2}^2 \overline{X_2}^2 \overline{Y_2}^2 \overline{Z_2}^2} \leq 0 \quad \widehat{g} := \det g = \frac{-4^3 \langle H_2 \rangle^3 \langle J_2 \rangle^3 \langle K_2 \rangle^3}{\langle W_2 \rangle^2 \langle X_2 \rangle^2 \langle Y_2 \rangle^2 \langle Z_2 \rangle^2} \leq 0$$

$$(13b) \quad \Gamma := \frac{\widehat{G}}{\widehat{g}} = \left( \frac{\langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle}{4\langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle \overline{W_2 X_2 Y_2 Z_2}} \right)^2$$

Cofactors of entries along the diagonals of the matrices have these forms:

$$(14a) \quad M_W = -4[A, B, C] = \frac{\langle W_2 \rangle^2}{\overline{X_2}^2 \overline{Y_2}^2 \overline{Z_2}^2}$$

$$(14b) \quad m_W = -4[d, e, f] = -16\overline{W_2}^2 \ddot{a}_2^2 \ddot{e}_2^2 \ddot{f}_2^2 = - \left( 4\overline{W_2} \frac{\langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle}{\langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle} \right)^2$$

while, via (105a), the remaining cofactors of  $G$  and  $g$  have these forms:

$$(15a) \quad M_{WX} = \ddot{a} \sqrt{M_W M_X} = (\ddot{a}_2^2 + \overline{a}_2^2) \sqrt{M_W M_X} = \frac{\langle J_2 \rangle \langle K_2 \rangle + 4\langle H_2 \rangle \llbracket H_2 Y_2 Z_2 \rrbracket}{\overline{W_2 X_2} \overline{Y_2}^2 \overline{Z_2}^2}$$

$$(15b) \quad m_{WX} = \ddot{D} \sqrt{m_W m_X} = \left( \ddot{H}_2 - \overline{W}_2 \ddot{X}_2 \right) \frac{4^2 \langle H_2 \rangle^2 \langle J_2 \rangle^2 \langle K_2 \rangle^2}{\langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle^2 \langle Z_2 \rangle^2}$$

One readily verifies identities<sup>7,8</sup> presented by Mednykh and Paskevich in [3], such as:

$$(16a) \quad \frac{M_W}{m_W} = \frac{M_X}{m_X} = \frac{M_Y}{m_Y} = \frac{M_Z}{m_Z} = -\Gamma$$

$$(16b) \quad \frac{\overline{AD}}{\overline{ad}} = \frac{\overline{BE}}{\overline{be}} = \frac{\overline{CF}}{\overline{cf}} = \sqrt{\Gamma}$$

$$(16c) \quad \frac{\ddot{A}\ddot{D} - \ddot{B}\ddot{E}}{\ddot{a}\ddot{d} - \ddot{b}\ddot{e}} = \frac{\ddot{B}\ddot{E} - \ddot{C}\ddot{F}}{\ddot{b}\ddot{e} - \ddot{c}\ddot{f}} = \frac{\ddot{C}\ddot{F} - \ddot{A}\ddot{D}}{\ddot{c}\ddot{f} - \ddot{a}\ddot{d}} = -\sqrt{\Gamma}$$

$$(16d) \quad \frac{\overline{\overline{(A \pm_1 D)} - (B \pm_2 E)}}{\overline{(a \pm_1 d) - (b \pm_2 e)}} = \frac{\overline{\overline{(B \pm_3 E)} - (C \pm_4 F)}}{\overline{(b \pm_3 e) - (c \pm_4 f)}} = \frac{\overline{\overline{(C \pm_5 F)} - (A \pm_6 D)}}{\overline{(c \pm_5 f) - (a \pm_6 d)}} = -\sqrt{\Gamma}$$

<sup>7</sup>(16a) This follows from our expressions for  $\Gamma$ ,  $M_W$ ,  $m_W$ . M&P’s equation (2) —prior to Lemma 1— asserts equality between the ratios and *positive*  $\Gamma$ ; M&P’s derivation includes factors such as (in our notation)  $\sqrt{m_W}$ , but, by (14b),  $m_W \leq 0$ . Someone among us has a sign error. (16b) M&P’s Theorem 2, which follows from (3a) and (12). (16c) M&P’s Theorem 4. (16d) M&P’s Corollary 1, which follows from (16b) and (16c). The “ $\pm_i$ ”s match for a given  $i$ , but are otherwise independent.

<sup>8</sup>M&P’s result relating altitude lengths to  $\Gamma$  appears to be in error; see the footnote for equation (34).

Observe that the last two of these may be expressed as

$$(17a) \quad \ddot{A}\ddot{D} \ell + \ddot{a}\ddot{d} L = \ddot{B}\ddot{E} \ell + \ddot{b}\ddot{e} L = \ddot{C}\ddot{F} \ell + \ddot{c}\ddot{f} L = \Lambda$$

$$(17b) \quad \begin{aligned} \overline{\overline{(A \pm_1 D)}} \ell + \overline{\overline{(a \pm_1 d)}} L &= \overline{\overline{(B \pm_2 E)}} \ell + \overline{\overline{(b \pm_2 e)}} L \\ &= \overline{\overline{(C \pm_3 F)}} \ell + \overline{\overline{(c \pm_3 f)}} L = \Lambda \end{aligned}$$

using these values that are invariant under tetrahedral symmetries:

$$\begin{aligned} \ell &:= 4 \langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle \overline{W_2 X_2 Y_2 Z_2} \\ L &:= \langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle \\ \Lambda &:= 2 \langle J_2 \rangle^2 \langle K_2 \rangle^2 + 2 \langle K_2 \rangle^2 \langle H_2 \rangle^2 + 2 \langle H_2 \rangle^2 \langle J_2 \rangle^2 \\ &\quad + 4 \langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle \left( \ddot{W}_2 \ddot{X}_2 \ddot{Y}_2 \ddot{Z}_2 - \ddot{H}_2 \ddot{J}_2 \ddot{K}_2 \right) \\ &\quad - \langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle \end{aligned}$$

The ratio  $L/\ell = \sqrt{\Gamma}$ , as well as pairs of opposing edges and dihedral angles, are invariant under fundamental *Regge* symmetries<sup>9</sup> (for instance,  $R_H$  fixes  $a, d, A, D, g, G$ ); consequently,  $\ell/\Lambda$  and  $L/\Lambda$ —and thus equations (17)—are also *Regge*-invariant.

In anticipation of later discussions of *Regge* symmetries, consider  $R_H$ , defined by

$$(18) \quad \begin{aligned} A' &= A & B' &= -B_2 + E_2 + C_2 + F_2 & C' &= B_2 + E_2 - C_2 + F_2 \\ D' &= D & E' &= B_2 - E_2 + C_2 + F_2 & F' &= B_2 + E_2 + C_2 - F_2 \end{aligned}$$

in a tetrahedron with corresponding edges  $a', b', c', d', e', f'$ , where—except for  $a' = a$  and  $d' = d$ —we *do not* assume that these lengths are linear combinations of the originals.

Combining select components of (17) gives

$$(19a) \quad \begin{aligned} \Lambda &= \frac{1}{2} \left( \overline{\overline{(B - E)}} + \overline{\overline{(C + F)}} \right) \ell + \frac{1}{2} \left( \overline{\overline{(b - e)}} + \overline{\overline{(c + f)}} \right) L \\ &= \overline{\overline{B'E'}} \ell + \overline{\overline{(-b_2 + e_2 + c_2 + f_2)}} \overline{\overline{(b_2 - e_2 + c_2 + f_2)}} L \end{aligned}$$

$$(19b) \quad \begin{aligned} 0 &= \frac{1}{2} \left( \overline{\overline{(B - E)}} - \overline{\overline{(C + F)}} \right) \ell + \frac{1}{2} \left( \overline{\overline{(b - e)}} - \overline{\overline{(c + f)}} \right) L \\ &= \overline{\overline{B'E'}} \ell - \overline{\overline{(-b_2 + e_2 + c_2 + f_2)}} \overline{\overline{(b_2 - e_2 + c_2 + f_2)}} L \end{aligned}$$

By (17a) and (16b), these are equal, respectively, to  $\overline{\overline{B'E'}} \ell + \overline{\overline{b'e'}} L$  and  $\overline{\overline{B'E'}} \ell - \overline{\overline{b'e'}} L$ . Therefore,

$$(20a) \quad \overline{\overline{b'e'}} = \overline{\overline{(-b_2 + e_2 + c_2 + f_2)}} \overline{\overline{(b_2 - e_2 + c_2 + f_2)}}$$

$$(20b) \quad \overline{\overline{b'e'}} = \overline{\overline{(-b_2 + e_2 + c_2 + f_2)}} \overline{\overline{(b_2 - e_2 + c_2 + f_2)}}$$

which implies, for some choices of  $\pm$ ,

$$(21) \quad \{ b', e' \} = \pm(-b_2 + e_2) + c_2 + f_2 \quad \text{likewise,} \quad \{ c', f' \} = b_2 + e_2 \pm (-c_2 + f_2)$$

<sup>9</sup>See Section 11.



Further,

$$(22a) \quad \overline{(b_2 \pm e_2)}^2 = \frac{\ell}{2L} \left( \frac{\Lambda + L}{\ell} - \overline{(B \pm E)} \right) \quad \overline{(b_2 \pm e_2)}^2 = \frac{\ell}{2L} \left( \frac{\Lambda - L}{\ell} - \overline{(B \pm E)} \right)$$

$$(22b) \quad \overline{(c_2 \pm f_2)}^2 = \frac{\ell}{2L} \left( \frac{\Lambda + L}{\ell} - \overline{(C \pm F)} \right) \quad \overline{(c_2 \pm f_2)}^2 = \frac{\ell}{2L} \left( \frac{\Lambda - L}{\ell} - \overline{(C \pm F)} \right)$$

so that

$$(23) \quad 2\sqrt{\Gamma} \sinh(b_2 + e_2 \pm (c_2 + f_2))$$

$$(24) = 2\sqrt{\Gamma} ( \sinh(b_2 + e_2) \cosh(c_2 + f_2) \pm \cosh(b_2 + e_2) \sinh(c_2 + f_2) )$$

$$(25) = \sqrt{(\Lambda_- - \overline{(B + E)}) (\Lambda_+ - \overline{(C + F)})} \pm \sqrt{(\Lambda_+ - \overline{(B + E)}) (\Lambda_- - \overline{(C + F)})}$$

$$(26) = \sqrt{(\Lambda_- - \overline{(C' + F')}) (\Lambda_+ - \overline{(B' + E')})} \pm \sqrt{(\Lambda_+ - \overline{(C' + F')}) (\Lambda_- - \overline{(B' + E')})}$$

$$(27) = \pm 2\sqrt{\Gamma} \sinh(b'_2 + e'_2 \pm (c'_2 + f'_2))$$

which implies

$$(28) \quad b + e + c + f = b' + e' + c' + f' \quad b + e - c - f = -b' - e' + c' + f'$$

so that

$$(29) \quad b + e = c' + f' \quad c + f = b' + e'$$

We can write<sup>10</sup>

$$(30a) \quad 2 \{ \bar{a}, \bar{d} \} = \sqrt{\left(\frac{\Lambda}{L} + 1 - \overline{(A+D)}\right) \left(\frac{\Lambda}{L} + 1 - \overline{(A-D)}\right)} \\ \pm \sqrt{\left(\frac{\Lambda}{L} - 1 - \overline{(A+D)}\right) \left(\frac{\Lambda}{L} - 1 - \overline{(A-D)}\right)}$$

$$(30b) \quad 2 \{ \bar{a}, \bar{d} \} = \sqrt{\left(\frac{\Lambda}{L} - 1 - \overline{(A+D)}\right) \left(\frac{\Lambda}{L} + 1 - \overline{(A-D)}\right)} \\ \pm \sqrt{\left(\frac{\Lambda}{L} + 1 - \overline{(A+D)}\right) \left(\frac{\Lambda}{L} - 1 - \overline{(A-D)}\right)}$$

$$(30c) \quad 2 \{ \bar{A}, \bar{D} \} = \sqrt{\left(\frac{\Lambda}{\ell} + 1 - \overline{(a+d)}\right) \left(\frac{\Lambda}{\ell} + 1 - \overline{(a-d)}\right)} \\ \pm \sqrt{\left(\frac{\Lambda}{\ell} - 1 - \overline{(a+d)}\right) \left(\frac{\Lambda}{\ell} - 1 - \overline{(a-d)}\right)}$$

$$(30d) \quad 2 \{ \bar{A}, \bar{D} \} = \sqrt{-\left(\frac{\Lambda}{\ell} + 1 - \overline{(a+d)}\right) \left(\frac{\Lambda}{\ell} - 1 - \overline{(a-d)}\right)} \\ \pm \sqrt{-\left(\frac{\Lambda}{\ell} - 1 - \overline{(a+d)}\right) \left(\frac{\Lambda}{\ell} + 1 - \overline{(a-d)}\right)}$$

We can often (as in Section 5) use the conjugation ratio to catalyze the conversion between equations in edge-related lower-case symbols ( $a, m_*, g$ , etc.) and ones in angle- or area-related upper-case symbols ( $A, M_*, G$ , etc.), effectively doubling the utility of each result. We call these *conjugate* relations.

For example, write  $P := M_W M_X M_Y M_Z$  and  $p := m_W m_X m_Y m_Z$ . Then

$$(31) \quad P = \left( \frac{\langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle}{\overline{W_2^3 X_2^3 Y_2^3 Z_2^3}} \right)^2 \quad p = \left( \frac{4^4 \langle H_2 \rangle^4 \langle J_2 \rangle^4 \langle K_2 \rangle^4 \overline{W_2 X_2 Y_2 Z_2}}{\langle W_2 \rangle^3 \langle X_2 \rangle^3 \langle Y_2 \rangle^3 \langle Z_2 \rangle^3} \right)^2$$

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<sup>10</sup>Full disclosure: I'm not *entirely* sure that the signs work out properly in all cases.

and, again, one can readily verify these conjugate identities<sup>11</sup> from [3, Proposition 5]:

$$(32) \quad \widehat{g}P = \widehat{G}^3 \leftrightarrow \widehat{G}p = \widehat{g}^3 \quad P^3p = \widehat{G}^8 \leftrightarrow p^3P = \widehat{g}^8$$

---

<sup>11</sup>Given the interplay of factors  $S := \overline{W_2X_2Y_2Z_2}$ ,  $T := \langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle$ ,  $U := 4 \langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle$ , one can show that any vanishing product of integer powers of  $G = -U/S^2$ ,  $g = -U^3/T^2$ ,  $P = T^2/S^6$ ,  $p = S^2U^8/T^6$ , can be written  $(\widehat{G}p/\widehat{g}^3)^\alpha (Pp^3/\widehat{g}^8)^\beta = 1$  for integers  $\alpha, \beta$ . Abusing terminology, the relations  $\widehat{G}p = \widehat{g}^3$  and  $Pp^3 = \widehat{g}^8$  have all possible relations of this kind in their *integer span*. Likewise, their conjugates. For one relation and its conjugate to span all relations in this way, however, requires non-integer exponents.

## 5. ALTITUDES AND PSEUDO-ALTITUDES

Let  $w, x, y, z$  be altitudes to respective faces  $W, X, Y, Z$ . We have conjugate relations<sup>12</sup>

$$(33a) \quad \overline{w}^2 M_W = \overline{x}^2 M_X = \overline{y}^2 M_Y = \overline{z}^2 M_Z = -\widehat{G}$$

$$(33b) \quad \overline{w}^2 m_W = \overline{x}^2 m_X = \overline{y}^2 m_Y = \overline{z}^2 m_Z = \widehat{g}$$

whence<sup>13</sup>

$$(34) \quad \overline{wxyz} = \frac{\widehat{G}^2}{\sqrt{M_W M_X M_Y M_Z}} = \frac{\widehat{g}^2}{\sqrt{m_W m_X m_Y m_Z}} = \frac{4^2 \langle H_2 \rangle^2 \langle J_2 \rangle^2 \langle K_2 \rangle^2}{W_2 X_2 Y_2 Z_2 \langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle}$$

By (15a) and (15b), we can write

$$(35a) \quad \frac{\overline{wx} M_{WX}}{\overline{a}} = \frac{\overline{wy} M_{WY}}{\overline{b}} = \frac{\overline{wz} M_{WZ}}{\overline{c}} = \frac{\overline{yz} M_{YZ}}{\overline{d}} = \frac{\overline{zx} M_{ZX}}{\overline{e}} = \frac{\overline{xy} M_{XY}}{\overline{f}} = -\widehat{G}$$

$$(35b) \quad \frac{\overline{wx} m_{WX}}{\overline{D}} = \frac{\overline{wy} m_{WY}}{\overline{E}} = \frac{\overline{wz} m_{WZ}}{\overline{F}} = \frac{\overline{yz} m_{YZ}}{\overline{A}} = \frac{\overline{zx} m_{ZX}}{\overline{B}} = \frac{\overline{xy} m_{XY}}{\overline{C}} = \widehat{g}$$

Also,

$$(36) \quad \overline{w} \overline{W_2} \langle W_2 \rangle = \overline{x} \overline{X_2} \langle X_2 \rangle = \overline{y} \overline{Y_2} \langle Y_2 \rangle = \overline{z} \overline{Z_2} \langle Z_2 \rangle = \frac{\langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle}{4 \langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle} \sqrt{\widehat{g}}$$

Let  $h, j, k$  be pseudo-altitudes<sup>14</sup> between respective edge pairs  $(a, d), (b, e), (c, f)$ ; and let  $\tau_h, \tau_j, \tau_k$  be the corresponding *twists* about those pseudo-altitudes. Equation (110a) from Appendix B gives these conjugate relations

$$(37a) \quad \overline{adh} \overline{\tau_h} = \overline{bej} \overline{\tau_j} = \overline{cfk} \overline{\tau_k} = \sqrt{-\widehat{g}}$$

$$(37b) \quad \overline{ADh} \overline{\tau_h} = \overline{BEj} \overline{\tau_j} = \overline{CFk} \overline{\tau_k} = \sqrt{-\widehat{G}}$$

Recall now from (104b) that  $m_*$  is a Heronic sine product; invoking the product's "side-angle-side" form (98), we see that equalities (33b) and (37a) have an identical structure, giving rise to a result that neatly accommodates both *standard* and *pseudo* elements.<sup>15,16</sup>

<sup>12</sup>Equation (33b) agrees with Mednykh and Pashkevich [3, Proposition 4(ii)], which appears in this note as equation (105a). The second set of equations arises from dividing the first set through by  $-\Gamma$ . See (16a).

<sup>13</sup>This *disagrees* with Mednykh and Paskevich [3, Proposition 5(iv)], which claims that the product is  $\widehat{G}^2/M_W/M_X/M_Y/M_Z = 1/\Gamma$ .

<sup>14</sup>A pair of skew lines in hyperbolic space admits an orthogonal transversal. (See [2].) A *pseudo-altitude* is such a transversal, joining lines through opposing edges of a tetrahedron. The *twist* is the (supplementarily ambiguous) angle between an edge and the plane of the opposing edge and their mutual pseudo-altitude.

<sup>15</sup>Mednykh and Pashkevich [3, Proposition 6] invoke the area-based form of the Heronic sine product (104b) to arrive at this alternative result

$$\sin(\text{half-face}) \cdot \cosh(\text{half-edge}) \cosh(\text{half-edge}) \cosh(\text{half-edge}) \cdot \sinh(\text{altitude}) = \sqrt{-\widehat{g}}/4$$

for any (standard) "face", the "edge"s surrounding it, and the "altitude" to it. No pseudo-elements appear.

<sup>16</sup>The analogous product for Euclidean tetrahedra is "edge  $\cdot$  edge  $\cdot$  sin(angle)  $\cdot$  altitude = 6  $\cdot$  volume", which is valid for both standard and pseudo elements.

**Theorem 5.1** (Law of “Side-Angle-Side-Altitude” Sines). *The product*

$$(38) \quad \sinh(\text{edge}) \cdot \sinh(\text{edge}) \cdot \sin(\text{angle}) \cdot \sinh(\text{altitude})$$

*is a metric invariant for hyperbolic tetrahedra, with value  $\sqrt{-\widehat{g}}$  for any two distinct “edge”s and the “angle” and “altitude” they determine.<sup>17</sup>*

Moving to cosines, we expand Appendix B’s equation (110b) into conjugate relations

$$(39a) \quad \left| \ddot{c}\ddot{f} - \ddot{b}\ddot{e} \right| = \overline{a\bar{d}}\ddot{h} |\ddot{\tau}_h| \quad \left| \ddot{a}\ddot{d} - \ddot{c}\ddot{f} \right| = \overline{b\bar{e}}\ddot{j} |\ddot{\tau}_j| \quad \left| \ddot{b}\ddot{e} - \ddot{a}\ddot{d} \right| = \overline{c\bar{f}}\ddot{k} |\ddot{\tau}_k|$$

(39b)

$$\left| \ddot{C}\ddot{F} - \ddot{B}\ddot{E} \right| = \overline{A\bar{D}}\ddot{h} |\ddot{\tau}_h| \quad \left| \ddot{A}\ddot{D} - \ddot{C}\ddot{F} \right| = \overline{B\bar{E}}\ddot{j} |\ddot{\tau}_j| \quad \left| \ddot{B}\ddot{E} - \ddot{A}\ddot{D} \right| = \overline{C\bar{F}}\ddot{k} |\ddot{\tau}_k|$$

Isolating either of  $\ddot{h}$  or  $\ddot{\tau}_h$  in (37a) and (39a) gives the same result; that is, we find that  $u = \ddot{h}^2$  and  $u = \ddot{\tau}_h^2$  are the two roots in each of these conjugate quadratic equations:

$$(40a) \quad u^2 \overline{a^2 d^2} - u \left( \overline{a^2 d^2} + (\ddot{c}\ddot{f} - \ddot{b}\ddot{e})^2 - \widehat{g} \right) + (\ddot{c}\ddot{f} - \ddot{b}\ddot{e})^2 = 0$$

$$(40b) \quad u^2 \overline{A^2 D^2} - u \left( \overline{A^2 D^2} + (\ddot{C}\ddot{F} - \ddot{B}\ddot{E})^2 - \widehat{G} \right) + (\ddot{C}\ddot{F} - \ddot{B}\ddot{E})^2 = 0$$

As  $\ddot{h}^2 \geq 1 \geq \ddot{\tau}_h^2$  for hyperbolic cosine  $\ddot{h}$  and circular cosine  $\ddot{\tau}_h$ , we can write specifically

$$(41a) \quad \ddot{h}^2 = \frac{\overline{a^2 d^2} + (\ddot{c}\ddot{f} - \ddot{b}\ddot{e})^2 - \widehat{g} + \delta}{2\overline{a^2 d^2}} = \frac{\overline{A^2 D^2} + (\ddot{C}\ddot{F} - \ddot{B}\ddot{E})^2 - \widehat{G} + \Delta}{2\overline{A^2 D^2}}$$

$$(41b) \quad \ddot{\tau}_h^2 = \frac{\overline{a^2 d^2} + (\ddot{c}\ddot{f} - \ddot{b}\ddot{e})^2 - \widehat{g} - \delta}{2\overline{a^2 d^2}} = \frac{\overline{A^2 D^2} + (\ddot{C}\ddot{F} - \ddot{B}\ddot{E})^2 - \widehat{G} - \Delta}{2\overline{A^2 D^2}}$$

for non-negative discriminants,  $\delta$  and  $\Delta$ , which a convenient imaginary unit allows us to express as Heronic products:

$$(41c) \quad \delta^2 = - \left[ 2\overline{a\bar{d}}, 2(\ddot{c}\ddot{f} - \ddot{b}\ddot{e}), 2i\sqrt{-\widehat{g}} \right] \quad \Delta^2 = - \left[ 2\overline{A\bar{D}}, 2(\ddot{C}\ddot{F} - \ddot{B}\ddot{E}), 2i\sqrt{-\widehat{G}} \right]$$

Note that the above implies

$$(42) \quad \ddot{h}^2 - \ddot{\tau}_h^2 = \frac{\sqrt{- \left[ 2\overline{a\bar{d}}, 2(\ddot{b}\ddot{e} - \ddot{c}\ddot{f}), 2i\sqrt{-\widehat{g}} \right]}}{\overline{a^2 d^2}} = \frac{\sqrt{- \left[ 2\overline{A\bar{D}}, 2(\ddot{B}\ddot{E} - \ddot{C}\ddot{F}), 2i\sqrt{-\widehat{G}} \right]}}{\overline{A^2 D^2}}$$

<sup>17</sup>For adjacent edges, the “angle” lies between them, and the “altitude” drops to the face containing them; for opposite edges, the “altitude” is the pseudo-altitude joining them, and the “angle” is the corresponding twist. Note that taking the sine of a twist renders the supplementary ambiguity of the twist moot.

## 6. ORTHOGONAL TWISTS AND PERFECT TETRAHEDRA

A right-angle twist about a pseudo-altitude is naturally called *orthogonal*. A tetrahedron is called *perfect*<sup>18</sup> when all three pairs of opposing edges are orthogonally twisted about their pseudo-altitudes. For example, a *right-corner* tetrahedron ( $A = B = C = \pi/2$ ) is orthogonal, as is a *regular* tetrahedron whose faces are equilateral triangles.

By (39a) and (39b), a perfect tetrahedron's opposite elements combine symmetrically:

$$(43) \quad \ddot{a}\ddot{d} = \ddot{b}\ddot{e} = \ddot{c}\ddot{f} \quad \ddot{A}\ddot{D} = \ddot{B}\ddot{E} = \ddot{C}\ddot{F}$$

In terms of Gram cofactors, we have

$$(44) \quad M_{WX}M_{YZ} = M_{WY}M_{ZX} = M_{WZ}M_{XY} \quad m_{WX} m_{YZ} = m_{WY} m_{ZX} = m_{WZ} m_{XY}$$

Using the Law of Cosines for Opposite Angles (1a) to re-write the dihedral cosines above, we derive this hedronometric characterization of perfection:

$$(45) \quad \ddot{H}_2 \langle H_2 \rangle = \ddot{J}_2 \langle J_2 \rangle = \ddot{K}_2 \langle K_2 \rangle$$

An orthogonal twist simplifies the quadratics in (40), providing concise expression of the length of the corresponding pseudo-altitude; for example,

$$(46) \quad \tau_h = \frac{\pi}{2} \quad \Longrightarrow \quad \bar{h} = \frac{\sqrt{-\hat{g}}}{\bar{a}\bar{d}} = \frac{\sqrt{-\hat{G}}}{AD}$$

The pseudo-altitudes of a perfect tetrahedron, then, have this symmetric property

$$(47) \quad \bar{h}^2 \bar{j}^2 \bar{k}^2 = \frac{-\hat{g}^3}{\bar{a}^2 \bar{b}^2 \bar{c}^2 \bar{d}^2 \bar{e}^2 \bar{f}^2} = \frac{-\hat{G}^3}{A^2 B^2 C^2 D^2 E^2 F^2} \\ = \frac{\langle H_2 \rangle^3 \langle J_2 \rangle^3 \langle K_2 \rangle^3}{[[H_2 W_2 X_2]][[H_2 Y_2 Z_2]][[J_2 W_2 Y_2]][[J_2 Z_2 X_2]][[K_2 W_2 Z_2]][[K_2 X_2 Y_2]]}$$

The dependencies of perfection reduce the degrees of freedom in a tetrahedron's elements from six to four. That is to say, up to symmetry, *a perfect tetrahedron is uniquely determined by the areas of its faces*. In theory, we can remove all references to pseudofaces in our formulas by solving for  $H, J, K$  in terms of  $W, X, Y, Z$  via the system of equations comprising (45) and (1b); in practice, the results seem symbolically intractable, as with this degree-12 polynomial equation for  $\ddot{H}_2$ :

$$(48a) \quad 0 = 16\ddot{H}_2^{12} - 64\ddot{H}_2^{11} s_X + 24\ddot{H}_2^{10} (4s_X^2 - 3) - 8\ddot{H}_2^9 (8s_X^3 - 7s_Y s_Z - 42s_X) \\ + \dots + s (s + s_Y^2) (s + s_Z^2) \left(1 - \ddot{W}_2^2 - \ddot{X}_2^2\right) \left(1 - \ddot{Y}_2^2 - \ddot{Z}_2^2\right)$$

where

$$(48b) \quad s := 1 - \ddot{W}_2^2 - \ddot{X}_2^2 - \ddot{Y}_2^2 - \ddot{Z}_2^2 - 4\ddot{W}_2 \ddot{Z}_2 \ddot{Y}_2 \ddot{Z}_2 \\ s_X := \ddot{W}_2 \ddot{X}_2 + \ddot{Y}_2 \ddot{Z}_2 \quad s_Y := \ddot{W}_2 \ddot{Y}_2 + \ddot{Z}_2 \ddot{X}_2 \quad s_Z := \ddot{W}_2 \ddot{Z}_2 + \ddot{X}_2 \ddot{Y}_2$$

<sup>18</sup>Also *orthocentric* or *orthogonal*. *Perfect* is my own term, coined decades ago, so I'll keep it.

This equation and its counterparts may be helpful numerically, but what their sprawling symbolic structure is trying to say about the tetrahedron remains a mystery.<sup>19</sup>

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<sup>19</sup>Much the same is true in Euclidean space, which has its own expansive polynomials for pseudoface areas, but at least perfect Euclidean tetrahedra admit a quartic formula for volume in terms of face areas.

## 7. PSEUDOFACES: SHADOWS AND THE SEARCH FOR SUBSTANCE

For a Euclidean tetrahedron, the pseudoface associated with pair of opposite edges is a shadow: the quadrilateral projection of the remaining edges into any plane parallel to that pair. Projection and parallelism are tricky concepts in hyperbolic space, so the appropriate analog of a pseudoface shadow is non-obvious. We consider here a seemingly-promising construction.

Take a tetrahedron having vertices  $P, Q, R, S$ , with  $a := |PQ|$  and  $d := |RS|$ . Suppose the pseudo-altitude  $h$  meets  $PQ$  and  $RS$  at  $M$  and  $N$ , respectively, and define  $p := |MP|$ ,  $q := |MQ|$ ,  $r := |NR|$ ,  $s := |NS|$ . Let  $O$  be the midpoint of the pseudo-altitude.

Emulating the notion of a shadowy projection into a plane parallel to  $a$  and  $d$ , let  $P', Q', R', S'$  be the feet of perpendiculars dropped from respective points  $P, Q, R, S$  into the plane perpendicular to  $h$  at  $O$ . Write  $p' := |OP'|$ ,  $q' := |OQ'|$ ,  $r' := |OR'|$ ,  $s' := |OS'|$ .

We consider the candidacy of  $\square P'R'Q'S'$  for the geometric realization of pseudoface  $H$ . For simplicity, we will assume that edges  $a$  and  $d$  are orthogonally twisted about pseudo-altitude  $h$ , and that  $p = r$  (so that also  $p' = r'$ ).

Observe that  $\square PMOP'$  is a Lambert quadrilateral with legs of length  $h_2$  and  $p'$ , and with a co-leg of length  $p$ . A little trigonometry gives

$$\widehat{p'} = \frac{\widehat{p}}{\widehat{h}_2} \quad \ddot{p}'^2 = \frac{\ddot{p}^2 \ddot{h}_2^2}{1 + \ddot{p}^2 \ddot{h}_2^2} \quad \overline{p'}^2 = \frac{\overline{p}^2}{1 + \ddot{p}^2 \ddot{h}_2^2}$$

Moreover,  $\triangle P'OR'$  is an isosceles right triangle with leg length  $p'$  and area given by

$$\sin(\triangle P'OR') = \frac{\overline{p'p'}}{1 + \ddot{p}'\ddot{p}'} = \frac{\overline{p}^2}{1 + \ddot{p}^2 \ddot{h}}$$

We consider three cases.

- **$M$  and  $N$  are midpoints of  $PQ$  and  $RS$ .** This tetrahedron is symmetric about pseudo-altitude  $h$ , and we have that  $p = q = r = s = a_2$  and  $\ddot{b} = \ddot{c} = \ddot{e} = \ddot{f} = \ddot{p}^2 \ddot{h} = \ddot{a}_2 \ddot{h}$ , and that  $\square P'R'Q'S'$  comprises four identical copies of  $\triangle P'OR'$ . Consequently,  $\sin(\frac{1}{4}\square P'R'Q'S') = \sin(\triangle P'OR') = \overline{a_2^2}/2/\overline{b_2^2} = \overline{H_4}$  by (9b). Our “mid-plane shadow” is a geometric realization of pseudoface  $H$ !
- **$M = Q$  and  $N = S$ .** This tetrahedron is an *orthoscheme*, with  $q = s = 0$  and  $p = a = d = r$ ; assign  $b := |PR|$ ,  $c := |PS|$ ,  $e := |QS| = h$ ,  $f := |QR|$ , so that  $\ddot{b} = \ddot{a}^2 \ddot{h}$  and  $\ddot{c} = \ddot{f} = \ddot{a} \ddot{h}$ . The points  $Q'$  and  $S'$  coincide at  $O$ , so that  $\triangle P'OR'$  is  $\square P'R'Q'S'$ , with area given by  $\sin(\triangle P'OR') = \overline{a^2}/2/\overline{b_2^2}$ . On the other hand, as the reader can verify,  $\overline{H}$  via (9) has an incompatible, far-less-concise expression, so that this “mid-plane shadow” is *not* a realization of pseudoface  $H$ .
- **$h = 0$ .** This tetrahedron is *degenerate*, with coplanar vertices. We can abandon the orthogonal twist condition, and the supposition  $p = r$ , but let us assume that edges  $a$  and  $d$  themselves—not merely the lines determined by them—intersect as diagonals of *convex* quadrilateral  $\square PRQS$ . (Non-intersecting circumstances are



handled similarly.) Then the quadrilateral is the union of the tetrahedron's faces in two ways:  $|\square PRQS| = Y + Z = W + X$ . But also,  $A = D = \pi$ , so that  $\ddot{H}_2 = \cos(Y_2 + Z_2) = \cos(W_2 + X_2)$ . The quadrilateral—arguably, the shadow of the degenerate tetrahedron into a plane “perpendicular” to pseudo-altitude  $h$ —realizes pseudoface  $H$ . Moreover, the *bow-tie* quadrilateral  $\square PSRQ$ , with diagonals (say)  $b$  and  $e$ , has area<sup>20</sup>  $|\square PSRQ| = |Z - X| = |W - Y|$ , whereas  $B = E = 0$  and  $\ddot{J}_2 = \cos(Z_2 - X_2) = \cos(W_2 - Y_2)$ ; the bow-tie realizes pseudoface  $J$ . Bow-tie  $\square PRSQ$ , likewise, realizes  $K$ . Note that, while the bow-ties may be considered *shadows* of the tetrahedron, and in a plane (degenerately) “parallel” to pairs of opposing edges, they are *not* in a plane perpendicular to the pseudo-altitudes corresponding to  $J$  and  $K$ .

The symmetric case demonstrates that a straightforward construction can realize a pseudoface as a “shadow”, and the degenerate case further correlates pseudofaces with shadows; the orthoscheme case, however, indicates that there is work yet to do in devising a general construction of pseudoface shadows, if indeed pseudofaces *are* shadows in general.

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<sup>20</sup>We take a bow-tie's area as the absolute value of the difference of its triangular regions, as is consistent with tracing the four edges of the quadrilateral in order, traversing the triangles in different orientations.

## 8. SPECIAL TETRAHEDRA

**8.1. Regular Tetrahedra** ( $A = B = C = D = E = F$ ). If the dihedral angles of a hyperbolic tetrahedron match (so that the figure is necessarily perfect, by (43)), then its side lengths match ( $a = b = c = d = e = f$ ), its face areas match ( $W = X = Y = Z$ ), and its pseudoface areas match ( $H = J = K$ ). Moreover, any one metric—angle measure  $A$ , side length  $a$ , face area  $X$ , pseudoface area  $H$ —completely determines the tetrahedron. A sampling of relations among these metrics follows.

The most straightforward route to relating  $A$  and  $X$  passes through the realms of hyperbolic and spherical trigonometry. We recall that, by the definition of hyperbolic area, the plane angle  $\theta$  at any vertex of our equilateral faces satisfies  $X = \pi - 3\theta$ , so that  $\theta = \pi_3 - X_3$ . The dihedral angle  $A$  relates to face angle  $\theta$  by the spherical Law of Cosines for Sides:

$$(49) \quad \ddot{A} = \frac{\ddot{\theta} - \ddot{\theta}\ddot{\theta}}{\ddot{\theta}\ddot{\theta}} = \frac{\ddot{\theta}}{1 + \ddot{\theta}} = \frac{\cos(\pi_3 - X_3)}{1 + \cos(\pi_3 - X_3)} = \frac{\ddot{U}_3}{1 + \ddot{U}_3}$$

where  $U := \pi - X$ . By the Law of Cosines for Opposite Angles (1a),

$$(50) \quad \ddot{H}_2 = \ddot{X}_2^2 + \overline{X}_2^2 \ddot{A} = \frac{1}{2} \left( 1 + 4\ddot{U}_3 - 4\ddot{U}_3^2 \right) \implies \overline{H}_4 = \frac{1}{2} \left( 2\ddot{U}_3 - 1 \right)$$

We can also derive

$$(51a) \quad \overline{X}_2^2 = \overline{H}_4^2 (3 + 2\overline{H}_4) \quad \ddot{X}_2^2 = (1 + \overline{H}_4)^2 (1 - 2\overline{H}_4)$$

$$(51b) \quad \langle X_2 \rangle = 4\overline{H}_4^3 (1 + \overline{H}_4) / \sqrt{1 - 2\overline{H}_4} \quad \langle H_2 \rangle = 4\overline{H}_4^3 (1 + \overline{H}_4)$$

$$(51c) \quad \llbracket H_2, X_2, X_2 \rrbracket = 2\overline{H}_4^4 (1 + \overline{H}_4)$$

$$(51d) \quad \widehat{a}_2^2 = \frac{1}{1 - 2\overline{H}_4} \quad \overline{a}_2^2 = \frac{2\overline{H}_4}{1 - 2\overline{H}_4} \quad \widehat{a}_2^2 = 2\overline{H}_4 = 2\ddot{U}_3 - 1 = \frac{3\ddot{A} - 1}{1 - \ddot{A}}$$

**8.2. Isohedral Right-Corner Tetrahedra** ( $A = B = C = \pi/2$ ,  $D = E = F$ ). Here, three areas match ( $X = Y = Z$ ), all pseudoface areas match ( $H = J = K$ ), and triples of side lengths match ( $a = b = c$  and  $d = e = f$ , with  $\ddot{d} = \ddot{a}^2$ ). The First Law of Cosines relates hypotenuse-face  $W$  and leg-face  $X$

$$(52) \quad \ddot{W}_2 = \ddot{X}_2^3 - \overline{X}_2^3 = \frac{1}{2} (2 + \overline{X}) \sqrt{1 - \overline{X}} \quad \overline{W}_2^2 = \frac{1}{4} \overline{X}^2 (3 + \overline{X})$$

but the spherical Law of Cosines applied to plane angle  $\theta := \pi_3 - W_3$  in hypotenuse  $W$  and (acute) plane angle  $\phi := \pi_4 - X_2$  in leg  $X$  gives something more direct:

$$(53) \quad 0 = \ddot{A} = \frac{\ddot{\theta} - \ddot{\phi}\ddot{\phi}}{\ddot{\phi}\ddot{\phi}} \implies \ddot{\phi}^2 = \ddot{\theta} \implies \overline{X} = 2(\overline{\pi_3 - W_3}) - 1$$

From the Law of Cosines for Opposite Angles (1a),

$$(54) \quad \ddot{X}_2^2 = \ddot{H}_2 = \ddot{W}_2 \ddot{X}_2 + \overline{W}_2 \overline{X}_2 \ddot{D}$$

so that

$$(55) \quad \ddot{D} = \frac{\ddot{T}_2}{\sqrt{1 + \ddot{T}_2^2}} = \frac{\sqrt{2\ddot{U}_3}}{2\ddot{U}_6} \quad \overline{D} = \frac{1}{\sqrt{1 + \ddot{T}_2^2}} = \frac{\sqrt{2}}{2\ddot{U}_6}$$

where  $T := \pi_2 - X$  and  $U := \pi - W$ . Also,

$$(56a) \quad \langle H_2 \rangle = \ddot{X}_2 \overline{X}_2^3 \quad \langle W_2 \rangle = \overline{X}_2^3 \quad \langle X_2 \rangle = \ddot{X}_2 \overline{X}_2^3 (\ddot{X}_2 - \overline{X}_2)$$

$$(56b) \quad \llbracket H_2 X_2 X_2 \rrbracket = \frac{1}{4} \overline{X}_2^4 \quad \llbracket H_2 W_2 X_2 \rrbracket = \frac{1}{2} \ddot{X}_2^2 \overline{X}_2^4$$

$$(56c) \quad \ddot{a}_2^2 = \frac{\ddot{X}_2}{\ddot{X}_2 - \overline{X}_2} \quad \overline{a}_2^2 = \frac{\overline{X}_2}{\ddot{X}_2 - \overline{X}_2} \quad \widehat{a}_2^2 = \widehat{X}_2$$

$$(56d) \quad \ddot{d}_2^2 = \frac{1}{1 - \overline{X}} \quad \overline{d}_2^2 = \frac{\overline{X}}{1 - \overline{X}} \quad \widehat{d}_2^2 = \overline{X} = 2\ddot{U}_3 - 1 = \frac{3\ddot{D}^2 - 1}{\overline{D}^2}$$

## 9. VOLUME

Volume seems the least-scrutable measurement in hyperbolic space, even for tetrahedra. The principal formula is a differential equation from Schläfli [6] in 1950:

$$(57) \quad dV = -a_2 dA - b_2 dB - c_2 dC - d_2 dD - e_2 dE - f_2 dF$$

Derevnin and Mednykh's 2005 formula [1] takes the form of a monolithic integral:

$$(58) \quad V = -\frac{1}{4} \int_{\alpha-\beta}^{\alpha+\beta} \log \frac{\cos \frac{A+B+C+\theta}{2} \cos \frac{A+E+F+\theta}{2} \cos \frac{D+B+F+\theta}{2} \cos \frac{D+E+C+\theta}{2}}{\sin \frac{A+D+B+E+\theta}{2} \sin \frac{B+E+C+F+\theta}{2} \sin \frac{C+F+A+D+\theta}{2} \sin \frac{\theta}{2}} d\theta$$

where

$$\alpha := \operatorname{atan} \frac{p_\alpha}{q_\alpha} \quad \beta := \operatorname{atan} \frac{p_\beta}{q_\beta}$$

and

$$\begin{aligned} p_\alpha &:= \sin(A+B+C+D+E+F) + \sin(A+D) + \sin(B+E) + \sin(C+F) \\ &\quad + \sin(D+E+F) + \sin(D+B+C) + \sin(A+E+C) + \sin(A+B+F) \\ q_\alpha &:= -(\cos(A+B+C+D+E+F) + \cos(A+D) + \cos(B+E) + \cos(C+F) \\ &\quad + \cos(D+E+F) + \cos(D+B+C) + \cos(A+E+C) + \cos(A+B+F)) \\ p_\beta &:= \sqrt{p_\alpha^2 + q_\alpha^2 - p_\beta^2} = 2\sqrt{-\det G} \\ q_\beta &:= 2(\sin A \sin D + \sin B \sin E + \sin C \sin F) \end{aligned}$$

We note that

$$(59) \quad \begin{aligned} p_\alpha^2 + q_\alpha^2 &= p_\beta^2 + q_\beta^2 \\ &= 8 \left( 1 + \overline{A\ddot{B}\ddot{C}} + \overline{A\ddot{E}\ddot{F}} + \overline{D\ddot{B}\ddot{F}} + \overline{D\ddot{E}\ddot{C}} \right. \\ &\quad \left. + \overline{A\ddot{D}\ddot{B}\ddot{E}} + \overline{B\ddot{E}\ddot{C}\ddot{F}} + \overline{C\ddot{F}\ddot{A}\ddot{D}} + \overline{ADBE} + \overline{BECE} + \overline{CFAD} \right) \end{aligned}$$

Also, with considerable trigonometric effort, one finds that the difference of the numerator and denominator in the logarithm's argument reduces thusly:

$$(60) \quad \cos(\cdot) \cos(\cdot) \cos(\cdot) \cos(\cdot) - \sin(\cdot) \sin(\cdot) \sin(\cdot) \sin(\cdot) = q_\beta - (p_\alpha \sin \theta + q_\alpha \cos \theta)$$

The values  $\theta = \alpha \pm \beta$  are roots of this expression and thus are also roots of the integrand.

Unfortunately —unlike in the Euclidean case<sup>21</sup>— these formula, in general, resist direct hedronometric re-parameterization in terms of face and pseudoface areas. (Below, we examine a couple of less-resistant special cases, and we discuss the complexity of the general

<sup>21</sup>The volume,  $V$ , of a Euclidean tetrahedron with faces  $W, X, Y, Z$  and pseudofaces  $H, J, K$  satisfies a formula bearing a striking resemblance to the Third Law of Cosines in hyperbolic space (1b).

$$\begin{aligned} 81V^4 &= 2W^2X^2Y^2 + 2W^2Y^2Z^2 + 2W^2Z^2X^2 + 2X^2Y^2Z^2 + H^2J^2K^2 \\ &\quad - H^2(W^2X^2 + Y^2Z^2) - J^2(W^2Y^2 + Z^2X^2) - K^2(W^2Z^2 + X^2Y^2) \end{aligned}$$

case.) For the most part, the best we can do currently is invoke the Law of Cosines for Opposite Angles (1a) to convert areas into dihedral angle measures, and substitute those measures into the Derevnin-Mednykh formula (58).

**9.1. Regular Tetrahedra** ( $A = B = C = D = E = F$ ). Since  $a = b = c = d = e = f$  and  $A = B = C = D = E = F$ , we have  $dV = -6 a_2 dA$ . We can express this differential in terms of angle  $A$ , face area  $X$ , or pseudoface area  $H$ :

$$(61a) \quad -\frac{dV}{6} = \operatorname{atanh} \sqrt{\frac{3\ddot{A} - 1}{1 - \ddot{A}}} dA = \operatorname{atanh} \sqrt{2\ddot{U}_3 - 1} \frac{dA}{dX} dX = \operatorname{atanh} \sqrt{2\overline{H}_4} \frac{dA}{dH} dH$$

$$(61b) \quad \frac{dA}{dX} = -\frac{\widehat{U}_6}{3\sqrt{2\ddot{U}_3 + 1}} \quad \frac{dA}{dH} = -\frac{\overline{L}_8}{2(3 + 2\ddot{L}_4)}$$

with  $U := \pi - X$  and  $L := 2\pi - H$ . So

$$(62a) \quad V = -6 \int_{\operatorname{acos}\frac{1}{3}}^A \operatorname{atanh} \sqrt{\frac{3 \cos \theta - 1}{1 - \cos \theta}} d\theta$$

$$(62b) \quad = 2 \int_0^X \frac{\tan(\pi_6 - \xi_6)}{\sqrt{2 \cos(\pi_3 - \xi_3) + 1}} \operatorname{atanh} \sqrt{2 \cos(\pi_3 - \xi_3) - 1} d\xi$$

$$(62c) \quad = 3 \int_0^H \frac{\sin(\pi_4 - \eta_8)}{3 + 2 \sin \eta_4} \operatorname{atanh} \sqrt{2 \sin \eta_4} d\eta$$

**9.2. Isohedral Right-Corner Tetrahedra** ( $A = B = C = \pi/2$ ,  $D = E = F$ ). Restricting ourselves (and our paths of integration) to the universe of right-corner tetrahedra with  $A = B = C = \pi/2$ , we can assert  $dA = dB = dC = 0$ ; then, with  $D = E = F$ , we have  $dD = dE = dF$ , so that  $dV = -3 d_2 dD$ . We can write:

$$(63a) \quad -\frac{dV}{3} = \operatorname{atanh} \frac{\sqrt{3\ddot{D}^2 - 1}}{\overline{D}} dD = \operatorname{atanh} \sqrt{\overline{X}} \frac{dD}{dX} dX = \operatorname{atanh} \sqrt{2\ddot{U}_3 - 1} \frac{dD}{dW} dW$$

$$(63b) \quad \frac{dD}{dX} = \frac{-\overline{T}_2}{4(1 + \ddot{T}_2^2)} \quad \frac{dD}{dW} = \frac{-\widehat{U}_6}{6\sqrt{\ddot{U}_3}}$$

with  $T := \pi_2 - X$  and  $U := \pi - W$ . So

$$(64a) \quad V = -3 \int_{\arccos \sqrt{\frac{1}{3}}}^D \operatorname{atanh} \frac{\sqrt{3 \cos^2 \theta - 1}}{\sin \theta} d\theta$$

$$(64b) \quad = \frac{3}{2} \int_0^X \frac{\sin(\pi_4 - \xi_2)}{1 + \cos^2(\pi_4 - \xi_2)} \operatorname{atanh} \sqrt{\sin \xi} d\xi$$

$$(64c) \quad = \frac{1}{2} \int_0^W \frac{\tan(\pi_6 - \omega_6)}{\sqrt{\cos(\pi_3 - \omega_3)}} \operatorname{atanh} \sqrt{2 \cos(\pi_3 - \omega_3) - 1} d\omega$$

**9.3. General Tetrahedra.** A key complication in deriving a face-based formula for tetrahedral volume is that our seven hedronometric parameters —faces areas  $W$ ,  $X$ ,  $Y$ ,  $Z$  and pseudoface areas  $H$ ,  $J$ ,  $K$ — aren't independent (and shouldn't be, as there are only *six* degrees of freedom in determining a tetrahedron). Nevertheless, the “differentialized” Law of Cosines for Opposite Angles (1a) converts the Schlafli angular differentials to hedronometric ones:

$$(65a) \quad 4\sqrt{[H_2 Y_2 Z_2]} dA = \overline{H_2} dH - \frac{\ddot{Z}_2 - \ddot{Y}_2 \ddot{H}_2}{\overline{Y_2}} dY - \frac{\ddot{Y}_2 - \ddot{Z}_2 \ddot{H}_2}{\overline{Z_2}} dZ$$

$$4\sqrt{[H_2 W_2 X_2]} dD = \overline{H_2} dH - \frac{\ddot{X}_2 - \ddot{W}_2 \ddot{H}_2}{\overline{W_2}} dW - \frac{\ddot{W}_2 - \ddot{X}_2 \ddot{H}_2}{\overline{X_2}} dX$$

while the differentialized Third Law of Cosines (1b) expresses the dependency among the hedronometric differentials:

$$(65b) \quad \langle W_2 \rangle \overline{W_2} dW + \langle X_2 \rangle \overline{X_2} dX + \langle Y_2 \rangle \overline{Y_2} dY + \langle Z_2 \rangle \overline{Z_2} dZ$$

$$= \langle H_2 \rangle \overline{H_2} dH + \langle J_2 \rangle \overline{J_2} dJ + \langle K_2 \rangle \overline{K_2} dK$$

While we hope for a monolithic integral in the spirit of Derevnin-Mednykh that respects the inherent symmetries of the face and pseudoface area parameters, for now we investigate an asymmetric solution.

**9.3.1. Using an auxiliary parameter.** Consider the universe of tetrahedra with constant areas  $W$ ,  $X$ ,  $Y$ ,  $Z$ ,  $H$  (and thus also constant dihedral angles  $A$  and  $D$ ), such that  $\overline{H_2} \neq 0$ .<sup>22</sup> Variable pseudoface areas  $J$  and  $K$  have a dependency via the Third Law of Cosines (1b). We can interpret the Law as a quadratic in  $\ddot{J}_2$  and  $\ddot{K}_2$  that describes an ellipse,<sup>23</sup> allowing us to parameterize the values in terms of a single (and remarkably well-suited) “polar” angle,

<sup>22</sup>At least one pseudoface  $P$  satisfies  $\overline{P_2} \neq 0$  in a non-degenerate tetrahedron. Without loss of generality, we take this pseudoface to be  $H$  (with a compatible labeling the tetrahedron's faces).

<sup>23</sup>The ellipse is rotated by  $\pi/4$  from the “ $\ddot{J}_2 \ddot{K}_2$ ” coordinate axes. Its center is  $(S_Y - \ddot{H}_2 S_Z, S_Z - \ddot{H}_2 S_Y) / \overline{H_2}^2$ , where  $S_Y := \ddot{W}_2 \ddot{Y}_2 + \ddot{Z}_2 \ddot{X}_2$  and  $S_Z := \ddot{W}_2 \ddot{Z}_2 + \ddot{X}_2 \ddot{Y}_2$ ; its radii are  $s \ddot{H}_4$  and  $s \overline{H_4}$ , where  $s := 4\sqrt{2} [H_2 W_2 X_2] [H_2 Y_2 Z_2] / \overline{H_2}^2$ .

$\theta$ . Translating and rotating the ellipse, we arrive at these relations, where we indicate dependence of  $J$  and  $K$  on  $\theta$  with a superscript:

$$(66a) \quad \begin{aligned} \ddot{J}_2^\theta \overline{H_2}^2 &= \dot{W}_2 \ddot{Y}_2 + \ddot{Z}_2 \ddot{X}_2 - \ddot{H}_2 \left( \dot{W}_2 \ddot{Z}_2 + \ddot{X}_2 \dot{Y}_2 \right) \\ &\quad + 4 \cos(\theta - H_4) \sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]} \end{aligned}$$

$$(66b) \quad \begin{aligned} \ddot{K}_2^\theta \overline{H_2}^2 &= \dot{W}_2 \ddot{Z}_2 + \ddot{X}_2 \dot{Y}_2 - \ddot{H}_2 \left( \dot{W}_2 \dot{Y}_2 + \ddot{Z}_2 \ddot{X}_2 \right) \\ &\quad - 4 \cos(\theta + H_4) \sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]} \end{aligned}$$

Corresponding formulas for  $\langle J_2^\theta \rangle$  and  $\langle K_2^\theta \rangle$  reduce nicely

$$(67a) \quad \langle J_2^\theta \rangle \overline{H_2} = 4 \sin(\theta + H_4) \sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]}$$

$$(67b) \quad \langle K_2^\theta \rangle \overline{H_2} = 4 \sin(\theta - H_4) \sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]}$$

and provide the insight that, given the non-negative nature of each other factor in the products, we must have  $\sin(\theta \pm H_4) \geq 0$ , so that we can take  $H_4 \leq \theta \leq \pi - H_4$ .<sup>24</sup>

Again using a superscript to denote dependence on  $\theta$ , we define

$$(68) \quad \langle H_2 \rangle^\theta := \ddot{H}_2 + \ddot{J}_2^\theta \ddot{K}_2^\theta - \dot{W}_2 \ddot{X}_2 - \dot{Y}_2 \ddot{Z}_2$$

and observe that  $\langle H_2 \rangle^\theta$  does not appear to admit insightful simplification. In particular, we have no clear way to identify degenerate tetrahedra for which  $\langle H_2 \rangle^\theta$  vanishes; such tetrahedra can restrict the range of viability of  $\theta$ .

The relations (67) provide compact representations of the trigonometric functions of  $\theta$ :<sup>25</sup>

$$(69a) \quad 4\overline{H_4} \sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]} = \overline{H_4} (\langle J_2^\theta \rangle + \langle K_2^\theta \rangle)$$

$$(69b) \quad 4\overline{H_4} \sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]} = \overline{H_4} (\langle J_2^\theta \rangle - \langle K_2^\theta \rangle)$$

$$(69c) \quad \widehat{\theta} = \widehat{H_4} \frac{\langle J_2^\theta \rangle + \langle K_2^\theta \rangle}{\langle J_2^\theta \rangle - \langle K_2^\theta \rangle}$$

With most face areas constant, the differentials of  $J^\theta$  and  $K^\theta$  are straightforward:<sup>26</sup>

$$(70a) \quad \overline{J_2^\theta} \overline{H_2}^2 dJ^\theta = 8 \sin(\theta - H_4) \sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]} d\theta$$

$$(70b) \quad \overline{K_2^\theta} \overline{H_2}^2 dK^\theta = -8 \sin(\theta + H_4) \sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]} d\theta$$

<sup>24</sup>The assumption  $\overline{H_2} \neq 0$  ensures that  $H_4 \neq \pi/2$ , so that this interval does not collapse to a point. Also, as indicated in (70), the lower (upper) bound corresponds to a point on the ellipse with a vertical (horizontal) tangent line. Furthermore, since the Gram determinant is proportional to  $\langle J_2 \rangle \langle K_2 \rangle$ , these endpoints correspond to degenerate tetrahedra.

<sup>25</sup>Squaring and adding the sine and cosine relations re-confirms equation (6).

<sup>26</sup>With (67), these are evidently consistent with the differentialized Third Law of Cosines (65b).

Moreover, the various angular differentials (as in (65a)) simplify. Of course,  $dA$  and  $dD$  vanish outright; otherwise, we have, for instance,

$$(70c) \quad dB = \frac{\overline{J_2^\theta} dJ^\theta}{4\sqrt{[[J_2^\theta Z_2 X_2]]}} = \frac{2\sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]}}{\overline{H_2}^2} \frac{\sin(\theta - H_4)}{\sqrt{[[J_2^\theta Z_2 X_2]]}} d\theta$$

We may therefore give this formula for the Schaffli volume differential (57):

$$(71a) \quad dV = \frac{\sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]}}{\overline{H_2}^2} \left( \begin{array}{c} -\sin(\theta - H_4) \left( \frac{b^\theta}{\sqrt{[[J_2^\theta Z_2 X_2]]}} + \frac{e^\theta}{\sqrt{[[J_2^\theta W_2 Y_2]]}} \right) \\ +\sin(\theta + H_4) \left( \frac{c^\theta}{\sqrt{[[K_2^\theta X_2 Y_2]]}} + \frac{f^\theta}{\sqrt{[[K_2^\theta W_2 Z_2]]}} \right) \end{array} \right) d\theta$$

for non-negative  $b^\theta, c^\theta, e^\theta, f^\theta$  given by

$$(71b) \quad \begin{aligned} b^\theta &= 2 \operatorname{atanh} \sqrt{4 \frac{[[J_2^\theta Z_2 X_2]]}{\langle H_2 \rangle^\theta} \frac{\sin(\theta + H_4)}{\sin(\theta - H_4)}} & e^\theta &= 2 \operatorname{atanh} \sqrt{4 \frac{[[J_2^\theta W_2 Y_2]]}{\langle H_2 \rangle^\theta} \frac{\sin(\theta + H_4)}{\sin(\theta - H_4)}} \\ c^\theta &= 2 \operatorname{atanh} \sqrt{4 \frac{[[K_2^\theta X_2 Y_2]]}{\langle H_2 \rangle^\theta} \frac{\sin(\theta - H_4)}{\sin(\theta + H_4)}} & f^\theta &= 2 \operatorname{atanh} \sqrt{4 \frac{[[K_2^\theta W_2 Z_2]]}{\langle H_2 \rangle^\theta} \frac{\sin(\theta - H_4)}{\sin(\theta + H_4)}} \end{aligned}$$

Now, note that our constant areas make the variable aspect of  $\widehat{G}$ , the tetrahedron's Gram determinant (see (13a)), proportional to  $\langle H_2 \rangle^\theta \langle J_2^\theta \rangle \langle K_2^\theta \rangle$ , which is in turn proportional to  $\langle H_2 \rangle^\theta \sin(\theta + H_4) \sin(\theta - H_4)$ . Consequently,  $\widehat{G}$  vanishes at the endpoints  $H_4$  and  $\pi - H_4$  of the bounds on  $\theta$  imposed after (67), and at the roots of  $\langle H_2 \rangle^\theta$ , which may lie between those endpoints.

Interpreting the above appropriately, we find that we can compute the volume of a tetrahedron via a hedronometric formula that integrates through our universe of tetrahedra with constant  $W, X, Y, Z, H$ , from a degenerate form (corresponding to  $\theta = H_4$ , or possibly some root  $\theta = \theta^*$  of  $\langle H_2 \rangle^\theta$ ) to our target (corresponding to the particular  $\theta$  satisfying (69), with  $J^\theta = J$  and  $K^\theta = K$ ).

**Theorem 9.1.** *A hyperbolic tetrahedron with face areas  $W, X, Y, Z$  and corresponding pseudoface areas  $H, J, K$  (with  $\overline{H_2} \neq 0$ ) has volume,  $V$ , given by*

$$(72) \quad V = \frac{\sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]}}{\overline{H_2}^2} \int_{\Phi}^{\Theta} \left( \begin{array}{c} -\sin(\theta - H_4) \left( \frac{b^\theta}{\sqrt{[[J_2^\theta Z_2 X_2]]}} + \frac{e^\theta}{\sqrt{[[J_2^\theta W_2 Y_2]]}} \right) \\ +\sin(\theta + H_4) \left( \frac{c^\theta}{\sqrt{[[K_2^\theta X_2 Y_2]]}} + \frac{f^\theta}{\sqrt{[[K_2^\theta W_2 Z_2]]}} \right) \end{array} \right) d\theta$$



with

- non-negative  $b^\theta, c^\theta, e^\theta, f^\theta$  from (71b);
- $H_4 \leq \Theta \leq \pi - H_4$ , such that  $\widehat{\Theta} = \widehat{H}_4 \frac{\langle J_2 \rangle + \langle K_2 \rangle}{\langle J_2 \rangle - \langle K_2 \rangle}$  (see (69)); and
- $\Phi$  the largest root of  $\langle H_2 \rangle^\theta \langle J_2^\theta \rangle \langle K_2^\theta \rangle = 0$  such that  $H_4 \leq \Phi \leq \Theta$ .

## 10. WHAT IS THE PYTHAGOREAN THEOREM FOR RIGHT-CORNER 4-SIMPLICES?

In Euclidean any-dimensional space, our understanding of the Pythagorean Theorem for right-corner simplices is complete: always, *the square of the content of the hypotenuse is equal to the sum of the squares of the contents of the legs*. In hyperbolic geometry, our knowledge seems embarrassingly limited:

$$\text{Dimension 2} \quad : \quad \cosh \text{hyp} = \cosh \text{leg}_1 \cosh \text{leg}_2$$

$$\text{Dimension 3} \quad : \quad \cos \frac{\text{hyp}}{2} = \cos \frac{\text{leg}_1}{2} \cos \frac{\text{leg}_2}{2} \cos \frac{\text{leg}_3}{2} - \sin \frac{\text{leg}_1}{2} \sin \frac{\text{leg}_2}{2} \sin \frac{\text{leg}_3}{2}$$

$$\text{Dimension 4+} \quad : \quad ??? = ???$$

Consider the case of a right-corner simplex in hyperbolic 4-space: as Section 9 indicates, any relation between the volume of the simplex's hypotenuse-tetrahedron and the volumes of its leg-tetrahedra involves comparing complicated integrals. Even the most-special of special cases (below) defies this author's attempts at resolution.

**10.1. Isosceles Right-Corner 4-Simplices.** Consider right-corner 4-simplex such that each of its four congruent legs are isohedral right-corner tetrahedra determined by dihedral angles  $(A, B, C, D, E, F) = (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, P, P, P)$ ; its hypotenuse is a regular tetrahedron with congruent dihedral angles  $(Q, Q, Q, Q, Q, Q)$ . One can verify that

$$(73) \quad \frac{1}{3} \leq \cos^2 P = \chi = \cos Q \leq \frac{1}{2}$$

for a parameter  $\chi$  that we use to write tantalizingly-similar formulas for the volume  $H$  of the hypotenuse (via (62a)) and volume  $L$  of each leg (via (64a)):

$$(74a) \quad H := -6 \int_{\arccos \frac{1}{3}}^{\arccos \chi} \operatorname{atanh} \sqrt{\frac{3 \cos \theta - 1}{1 - \cos \theta}} d\theta = 6 \int_{\frac{1}{3}}^{\chi} \frac{1}{\sqrt{1-t^2}} \operatorname{atanh} \sqrt{\frac{3t-1}{1-t}} dt$$

$$(74b) \quad L := -3 \int_{\arccos \sqrt{\frac{1}{3}}}^{\arccos \sqrt{\chi}} \operatorname{atanh} \sqrt{\frac{3 \cos^2 \theta - 1}{1 - \cos^2 \theta}} d\theta = \frac{3}{2} \int_{\frac{1}{3}}^{\chi} \frac{1}{\sqrt{t(1-t)}} \operatorname{atanh} \sqrt{\frac{3t-1}{1-t}} dt$$

Although Pythagoras beckons, a connection between these volumes remains elusive. Indeed, the rather exotic—but not unfamiliar<sup>27</sup>—values for  $\chi = 1/2$  (defining a simplex with a quadrupally-asymptotic hypotenuse and triply-asymptotic legs)

$$(75) \quad H^* := \sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{\pi k}{3} = 1.01494 \dots \quad L^* := \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{\pi k}{2} = 0.45798 \dots$$

suggest that the corresponding Pythagorean Theorem is quite non-trivial.

<sup>27</sup> $L^*$  is half of Catalan's constant, and  $H^*$  is known, e.g., as the maximum of the Clausen function,  $\text{Cl}_2$ .

## 11. REGGE SYMMETRIES

A *Regge symmetry* is an operation for deriving a strongly-related counterpart to a given tetrahedron. Such a symmetry is defined by what it does to the given tetrahedron's dihedral angles; for instance,

$$(76a) \quad R_H(A, B, C, D, E, F) := (A', B', C', D', E', F')$$

where

$$(76b) \quad \begin{aligned} A' &:= A & B' &:= -B_2 + C_2 + E_2 + F_2 & C' &:= B_2 - C_2 + E_2 + F_2 \\ D' &:= D & E' &:= B_2 + C_2 - E_2 + F_2 & F' &:= B_2 + C_2 + E_2 - F_2 \end{aligned}$$

Using the pseudoface subscript  $H$  emphasizes the dihedral angles fixed by the operation. The set of three fundamental Regge operations — $R_H$ ,  $R_J$ ,  $R_K$ — along with element permutations corresponding to the standard symmetries of the tetrahedron, generate the 144-element group of *Regge symmetries*, isomorphic to  $S_3 \times S_4$ .

The “strongly-related” nature of the derived tetrahedron involves a collection of matching (or proportional) metrics, and we'll outline some hedronometric instances below. The most significant aspect of this strong relation is that *all Regge-symmetric tetrahedra have the same volume*, which is clear from the Derevnin-Mednykh volume formula (58); in fact, Mohanty [4] demonstrates a much stronger result via explicit construction:

**Theorem 11.1** (Mohanty). *Any two Regge-symmetric tetrahedra are “scissors congruent”.*

In what follows, we survey how the particular Regge symmetry  $R_H$  affects various tetrahedral elements covered throughout this note. As with the dihedral angles, we append a prime ( $\prime$ ) to elements related to a tetrahedron's image under  $R_H$ .

**11.1. Angles, Edges, and Conjugation.** Regge symmetry  $R_H$  is defined to preserve dihedral angles  $A$  and  $D$ . Straightforward manipulation of a few trigonometric expressions shows that it also preserves determinant  $\widehat{G}$ , as well as cofactors  $M_{WX}$  and  $M_{YZ}$ , of the Gram matrix.

$$(77) \quad \widehat{G}' = \widehat{G} \quad M_{W'X'} = M_{WX} \quad M_{Y'Z'} = M_{YZ}$$

The symmetry preserves certain products of Heronic products:

$$(78) \quad \begin{aligned} \llbracket A'B'C' \rrbracket \llbracket A'E'F' \rrbracket &= \llbracket ABC \rrbracket \llbracket AEF \rrbracket & \llbracket A'B'C' \rrbracket \llbracket A'E'F' \rrbracket &= \llbracket ABC \rrbracket \llbracket AEF \rrbracket \\ \llbracket D'B'F' \rrbracket \llbracket D'E'C' \rrbracket &= \llbracket DBF \rrbracket \llbracket DEC \rrbracket & \llbracket D'B'F' \rrbracket \llbracket D'E'C' \rrbracket &= \llbracket DBF \rrbracket \llbracket DEC \rrbracket \end{aligned}$$

and thus also preserves products of Gram cofactors involving such products:

$$(79) \quad M_{W'}M_{X'} = M_W M_X \quad M_{Y'}M_{Z'} = M_Y M_Z$$

In light of this and (105a), we conclude that the symmetry preserves edge lengths  $a$  and  $d$ .

$$(80) \quad a' = a \quad d' = d$$

With  $A, D, a, d$  unchanged, (16b) implies that the conjugation ratio,  $\Gamma$ , is also unchanged

$$(81) \quad \Gamma' = \Gamma$$

which invites numerous conjugate equalities:<sup>28</sup>

$$(82a) \quad \widehat{g}' = \widehat{g} \quad m_{W'X'} = m_{WX} \quad m_{Y'Z'} = m_{YZ} \quad \Lambda'/\ell' = \Lambda/\ell$$

$$(82b) \quad m_{W'}m_{X'} = m_Wm_X \quad m_{Y'}m_{Z'} = m_Ym_Z$$

In fact, one can show<sup>29</sup> the ultimate conjugate result: that Regge symmetry  $R_H$  can be equivalently defined in terms of edge-lengths instead of dihedral angle-measures.

$$(83) \quad \begin{aligned} a' &:= a & b' &:= -b_2 + c_2 + e_2 + f_2 & c' &:= b_2 - c_2 + e_2 + f_2 \\ d' &:= d & e' &:= b_2 + c_2 - e_2 + f_2 & f' &:= b_2 + c_2 + e_2 - f_2 \end{aligned}$$

**11.2. Altitudes, Pseudo-altitudes, and Twists.** Equation (105a) gives us that certain products of sines of altitudes are unchanged by Regge symmetry  $R_H$ :

$$(84) \quad \overline{w'x'} = \overline{wx} \quad \overline{y'z'} = \overline{yz}$$

Moreover, by (37a), the symmetry preserves the product of the sines of the pseudo-altitude and twist corresponding to the symmetry's associated pair of opposing edges:<sup>30</sup>

$$(85) \quad \overline{h'\tau'_h} = \overline{h\tau_h}$$

Given the Regge-fixed nature of  $\widehat{g}$ , we find that the universality of Theorem 5.1's "Side-Angle-Side-Altitude" sine product extends across all Regge-symmetric tetrahedra (not merely images of  $R_H$ ):

**Corollary 11.2** (Law of "Side-Angle-Side-Altitude" Sines for Regge-Symmetric Tetrahedra). *The product*

$$(86) \quad \sinh(\text{edge}) \cdot \sinh(\text{edge}) \cdot \sin(\text{angle}) \cdot \sinh(\text{altitude})$$

*is a metric invariant across each family<sup>31</sup> of Regge-symmetric hyperbolic tetrahedra, with value  $\sqrt{-\widehat{g}}$  for any two distinct "edge"s, and the "angle" and "altitude" they determine, within a particular member of that family.*

<sup>28</sup>The last item in (82a) refers to the tetrahedral invariants  $\ell$  and  $\Lambda$  introduced in (17).

<sup>29</sup>Certainly, the substitutions in (83) satisfy the relations (82); however, that the relations imply the substitutions is not immediately obvious. Equations (??) and (??) are symmetric in pairs  $\{b', e'\}$  and  $\{c', f'\}$ , and cannot by themselves distinguish between these values; at best, they imply

$$\{b', e'\} = \pm(-b_2 + e_2) + c_2 + f_2 \quad \{c', f'\} = b_2 + e_2 \pm (-c_2 + f_2)$$

Relation (??) (or (??)) breaks some symbolic symmetry, and we can deduce that  $b'$ 's " $\pm$ " matches  $c'$ 's (and thus  $e'$ 's matches  $f'$ 's). To resolve the final sign ambiguity, one can verify that  $b+b' = c+c' (= e+e' = f+f')$ ; one computationally-arduous strategy is to show that  $\tanh(b+b') - \tanh(c+c')$  vanishes.

<sup>30</sup>However, since  $\ddot{B}'\ddot{E}' - \ddot{C}'\ddot{F}' = \overline{BE} - \overline{CF}$ , equation (39b) indicates that the product of the cosines of these values is not—and hence the values themselves are not—in general, preserved.

<sup>31</sup>That is, each collection of all images of a given tetrahedron under the elements of the group of Regge symmetries.

While on the topic of pseudo-altitudes and twists, we note this consequence of (41) in passing:

$$\begin{aligned}
 (87) \quad \ddot{h}^2 + \ddot{\tau}_h^2 - \left( \dot{h}'^2 + \dot{\tau}_h'^2 \right) &= \frac{\left( \ddot{C}\ddot{F} - \ddot{B}\ddot{E} \right)^2 - \left( \ddot{C}'\ddot{F}' - \ddot{B}'\ddot{E}' \right)^2}{\overline{A}^2 \overline{D}^2} \\
 &= \frac{\left( \ddot{C}\ddot{F} - \ddot{B}\ddot{E} \right)^2 - \left( \overline{C}\overline{F} - \overline{B}\overline{E} \right)^2}{\overline{A}^2 \overline{D}^2} \\
 &= 4 \frac{\sin(B_2 + C_2 + E_2 + F_2) \sin(B_2 + C_2 - E_2 - F_2)}{\overline{A}^2 \overline{D}^2} \\
 &\quad \cdot \sin(B_2 - C_2 + E_2 - F_2) \sin(B_2 - C_2 - E_2 + F_2) \\
 &= 4 \frac{\sin(b_2 + c_2 + e_2 + f_2) \sin(b_2 + c_2 - e_2 - f_2)}{\overline{a}^2 \overline{d}^2} \\
 &\quad \cdot \sin(b_2 - c_2 + e_2 - f_2) \sin(b_2 - c_2 - e_2 + f_2)
 \end{aligned}$$

**11.3. Hedronometric Elements and the Regge Ratio.** A Regge symmetry preserves tetrahedral volume, and it scrambles both dihedral angles and edge lengths via simple linear combinations. A few numerical experiments are enough to confirm that the effects on face areas are considerably less straightforward, and the introduction of pseudoface areas don't seem to help in this regard. (It certainly isn't the case that the areas of images of faces and pseudofaces are linear combinations of the originals.) That isn't to say that Regge symmetries have hopelessly-complicated effects on hedronometric elements; indeed not: we have a dizzying array of succinct relations among such elements. To get at them, we note a few hedronometric ratios that arise<sup>32</sup> from equalities involving angles and edges

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<sup>32</sup>Via (11), (13a), and (14a).

of  $R_H$ -related tetrahedra.

$$(88a) \quad \ddot{a}'_2 = \ddot{a}_2, \ddot{d}'_2 = \ddot{d}_2 \implies \frac{\langle J'_2 \rangle \langle K'_2 \rangle}{\langle J_2 \rangle \langle K_2 \rangle} = \frac{\langle W'_2 \rangle \langle X'_2 \rangle}{\langle W_2 \rangle \langle X_2 \rangle} = \frac{\langle Y'_2 \rangle \langle Z'_2 \rangle}{\langle Y_2 \rangle \langle Z_2 \rangle}$$

$$(88b) \quad \overline{a}'_2 = \overline{a}_2 \implies \frac{\langle H'_2 \rangle \llbracket H'_2 Y'_2 Z'_2 \rrbracket}{\langle H_2 \rangle \llbracket H_2 Y_2 Z_2 \rrbracket} = \frac{\langle W'_2 \rangle \langle X'_2 \rangle}{\langle W_2 \rangle \langle X_2 \rangle}$$

$$(88c) \quad \overline{d}'_2 = \overline{d}_2 \implies \frac{\langle H'_2 \rangle \llbracket H'_2 W'_2 X'_2 \rrbracket}{\langle H_2 \rangle \llbracket H_2 W_2 X_2 \rrbracket} = \frac{\langle Y'_2 \rangle \langle Z'_2 \rangle}{\langle Y_2 \rangle \langle Z_2 \rangle}$$

$$(88d) \quad \widehat{G}' = \widehat{G} \implies \frac{\langle H'_2 \rangle \langle J'_2 \rangle \langle K'_2 \rangle}{\langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle} = \left( \frac{\overline{W'_2 X'_2 Y'_2 Z'_2}}{\overline{W_2 X_2 Y_2 Z_2}} \right)^2$$

$$(88e) \quad \widehat{g}' = \widehat{g} \implies \left( \frac{\langle H'_2 \rangle \langle J'_2 \rangle \langle K'_2 \rangle}{\langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle} \right)^3 = \left( \frac{\langle W'_2 \rangle \langle X'_2 \rangle \langle Y'_2 \rangle \langle Z'_2 \rangle}{\langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle} \right)^2$$

$$(88f) \quad M_{W'} M_{X'} = M_W M_X \implies \frac{\langle W'_2 \rangle \langle X'_2 \rangle}{\langle W_2 \rangle \langle X_2 \rangle} = \frac{\overline{W'_2 X'_2 Y'_2 Z'_2}^2}{\overline{W_2 X_2 Y_2 Z_2}^2}$$

$$(88g) \quad M_{Y'} M_{Z'} = M_Y M_Z \implies \frac{\langle Y'_2 \rangle \langle Z'_2 \rangle}{\langle Y_2 \rangle \langle Z_2 \rangle} = \frac{\overline{W'_2 X'_2 Y'_2 Z'_2}^2}{\overline{W_2 X_2 Y_2 Z_2}^2}$$

We can distill the above to equalities involving a key ratio,  $\rho_H := \langle H'_2 \rangle / \langle H_2 \rangle$ :<sup>33,34</sup>

$$(89a) \quad \rho_H := \frac{\langle H'_2 \rangle}{\langle H_2 \rangle} = \frac{\overline{W'_2 X'_2}}{\overline{W_2 X_2}} = \frac{\overline{Y'_2 Z'_2}}{\overline{Y_2 Z_2}} = \frac{\ddot{H}'_2 - \ddot{W}'_2 \ddot{X}'_2}{\ddot{H}_2 - \ddot{W}_2 \ddot{X}_2} = \frac{\ddot{H}'_2 - \ddot{Y}'_2 \ddot{Z}'_2}{\ddot{H}_2 - \ddot{Y}_2 \ddot{Z}_2} = \frac{\ddot{H}'_2 - \ddot{J}'_2 \ddot{K}'_2}{\ddot{H}_2 - \ddot{J}_2 \ddot{K}_2}$$

$$(89b) \quad \rho_H^2 = \frac{\llbracket H'_2 W'_2 X'_2 \rrbracket}{\llbracket H_2 W_2 X_2 \rrbracket} = \frac{\llbracket H'_2 Y'_2 Z'_2 \rrbracket}{\llbracket H_2 Y_2 Z_2 \rrbracket}$$

$$(89c) \quad \rho_H^3 = \frac{\langle J'_2 \rangle \langle K'_2 \rangle}{\langle J_2 \rangle \langle K_2 \rangle} = \frac{\langle W'_2 \rangle \langle X'_2 \rangle}{\langle W_2 \rangle \langle X_2 \rangle} = \frac{\langle Y'_2 \rangle \langle Z'_2 \rangle}{\langle Y_2 \rangle \langle Z_2 \rangle}$$

$$(89d) \quad \frac{1}{\rho_H} = \frac{\ddot{b}'_2 \ddot{e}'_2 \ddot{c}'_2 \ddot{f}'_2}{\ddot{b}_2 \ddot{e}_2 \ddot{c}_2 \ddot{f}_2}$$

<sup>33</sup>For instance

$$\left( \frac{\overline{W'_2 X'_2 Y'_2 Z'_2}}{\overline{W_2 X_2 Y_2 Z_2}} \right)^2 = \frac{\langle H'_2 \rangle \langle J'_2 \rangle \langle K'_2 \rangle}{\langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle} = \frac{\langle H'_2 \rangle \langle W'_2 \rangle \langle X'_2 \rangle}{\langle H_2 \rangle \langle W_2 \rangle \langle X_2 \rangle} = \frac{\langle H'_2 \rangle \overline{W'_2 X'_2 Y'_2 Z'_2}^2}{\langle H_2 \rangle \overline{W_2 X_2 Y_2 Z_2}^2} \implies \frac{\langle H'_2 \rangle}{\langle H_2 \rangle} = \frac{\overline{W'_2 X'_2}}{\overline{W_2 X_2}}$$

<sup>34</sup>The fourth and fifth components of (a) follow from the Law of Cosines for Opposite Angles (1a).

A little extra work reveals many additional relations:<sup>35</sup>

$$\begin{aligned}
 (89e) \quad \rho_H &= \left( \frac{\ddot{H}'_4}{\ddot{H}_4} \right)^2 = \frac{\overline{J'_2 K'_2}}{\overline{J_2 K_2}} = \frac{1 + \overline{W'_2 X'_2}}{1 + \overline{W_2 X_2}} = \frac{1 + \overline{Y'_2 Z'_2}}{1 + \overline{Y_2 Z_2}} = \frac{1 + \overline{J'_2 K'_2}}{1 + \overline{J_2 K_2}} \\
 &= \frac{\overline{W'_2} \sqrt{\overline{M'_X}} + \overline{X'_2} \sqrt{\overline{M'_W}}}{\overline{W_2} \sqrt{\overline{M_X}} + \overline{X_2} \sqrt{\overline{M_W}}} = \frac{\overline{Y'_2} \sqrt{\overline{M'_Z}} + \overline{Z'_2} \sqrt{\overline{M'_Y}}}{\overline{Y_2} \sqrt{\overline{M_Z}} + \overline{Z_2} \sqrt{\overline{M_Y}}} \\
 (89f) \quad \rho_H^2 &= \frac{\langle \overline{W'_2} \rangle + \langle \overline{X'_2} \rangle}{\langle \overline{W_2} \rangle + \langle \overline{X_2} \rangle} = \frac{\langle \overline{Y'_2} \rangle + \langle \overline{Z'_2} \rangle}{\langle \overline{Y_2} \rangle + \langle \overline{Z_2} \rangle} = \frac{\langle \overline{J'_2} \rangle + \langle \overline{K'_2} \rangle}{\langle \overline{J_2} \rangle + \langle \overline{K_2} \rangle} = \frac{\llbracket \overline{H'_2 J'_2 K'_2} \rrbracket}{\llbracket \overline{H_2 J_2 K_2} \rrbracket}
 \end{aligned}$$

Picking out some key components,

$$(90) \quad \left( \frac{\ddot{H}'_4}{\ddot{H}_4} \right)^4 = \rho_H^2 = \frac{\llbracket \overline{H'_2 J'_2 K'_2} \rrbracket}{\llbracket \overline{H_2 J_2 K_2} \rrbracket}$$

we observe that left-hand side isolates  $R_H$ 's *preferred* pseudoface,  $H$ , whereas the right-hand side shows no preference whatsoever. That is, the right-hand side serves just as well to determine  $\rho_J^2$  and  $\rho_K^2$ . This realization prompts the following definition.<sup>36</sup>

**Definition 11.3.** *For a transformation  $R$  in the group of Regge symmetries, the Regge ratio,  $\rho_R \geq 0$ , is defined by*

$$(91) \quad \rho_R := \frac{\llbracket \overline{H'_2 J'_2 K'_2} \rrbracket}{\llbracket \overline{H_2 J_2 K_2} \rrbracket}$$

where  $H', J', K'$  are respective images of pseudofaces  $H, J, K$  under  $R$ .

We immediately note

$$(92) \quad \rho_T = 1 \quad \rho_{R \circ S} = \rho_R \cdot \rho_S$$

for standard tetrahedral symmetry  $T$  (under which  $(H', J', K')$  is simply a permutation of  $(H, J, K)$ ), and for any composition of elements  $R$  and  $S$  from  $\mathbf{R}$ . As we have shown above,  $\rho_R$  for  $R \in \{R_H, R_J, R_K\}$  provides the value of key ratios among many symmetry-preferring expressions; however, the significance of  $\rho_R$  for an arbitrary Regge symmetry—for instance,  $R := R_J \circ R_H$ —is not (yet) clear.

**11.4. More Regge ratio identities.** Further gleanings from the glut of Regge ratio identities:

$$(93a) \quad \rho_H = \frac{\left( \frac{\overline{W'_4 \pm X'_4}}{\overline{W_4 \pm X_4}} \right)^2}{\left( \frac{\overline{Y'_4 \pm Z'_4}}{\overline{Y_4 \pm Z_4}} \right)^2} = \frac{\left( \frac{\overline{Y'_4 \pm Z'_4}}{\overline{Y_4 \pm Z_4}} \right)^2}{\left( \frac{\overline{J'_4 \pm K'_4}}{\overline{J_4 \pm K_4}} \right)^2} = \frac{\left( \frac{\overline{J'_4 \pm K'_4}}{\overline{J_4 \pm K_4}} \right)^2}{\left( \frac{\overline{J_4 \pm K_4}}{\overline{J_4 \pm K_4}} \right)^2}$$

<sup>35</sup>Together with the preceding relations, these indicate a certain striking “interchangeability” of pairs  $(W, X)$ ,  $(Y, Z)$ ,  $(J, K)$  relative to  $\rho_H$ . (Indeed, many of the ratios depicted were originally guessed by replacing one pair with another in an existing proportion.) There must be a formulation of  $\rho_H$  that makes this observation both obvious and meaningful.

<sup>36</sup>I admit to abusing my own notation, writing “ $\rho_H$ ” for what I propose should be written “ $\rho_{R_H}$ ”.

It turns out that we can unambiguously take the square root.<sup>37</sup>

$$(93b) \quad \sqrt{\rho_H} = \frac{\left(\overline{W_4' \pm X_4'}\right)}{\left(\overline{W_4 \pm X_4}\right)} = \frac{\left(\overline{Y_4' \pm Z_4'}\right)}{\left(\overline{Y_4 \pm Z_4}\right)} = \frac{\left(\overline{J_4' \pm K_4'}\right)}{\left(\overline{J_4 \pm K_4}\right)}$$

The sum and product of each sum and difference pair give us these relations:

$$(94a) \quad \sqrt{\rho_H} = \frac{\overline{W_4' X_4'}}{\overline{W_4 X_4}} = \frac{\overline{Y_4' Z_4'}}{\overline{Y_4 Z_4}} = \frac{\overline{J_4' K_4'}}{\overline{J_4 K_4}}$$

$$(94b) \quad \rho_H = \frac{\overline{W_2' + X_2'}}{\overline{W_2 + X_2}} = \frac{\overline{Y_2' + Z_2'}}{\overline{Y_2 + Z_2}} = \frac{\overline{J_2' + K_2'}}{\overline{J_2 + K_2}}$$

Observe that we can use these equations to solve for (pseudo-)face pairs:

$$(95) \quad \begin{aligned} \left\{ \overline{W_2'}, \overline{X_2'} \right\} &= \frac{\rho_H}{2} \left( \overline{W_2 + X_2} \right) \pm \sqrt{\left( 1 - \rho_H \left( \overline{W_4 + X_4} \right)^2 \right) \left( 1 - \rho_H \left( \overline{W_4 - X_4} \right)^2 \right)} \\ \left\{ \overline{Y_2'}, \overline{Z_2'} \right\} &= \frac{\rho_H}{2} \left( \overline{Y_2 + Z_2} \right) \pm \sqrt{\left( 1 - \rho_H \left( \overline{Y_4 + Z_4} \right)^2 \right) \left( 1 - \rho_H \left( \overline{Y_4 - Z_4} \right)^2 \right)} \\ \left\{ \overline{J_2'}, \overline{K_2'} \right\} &= \frac{\rho_H}{2} \left( \overline{J_2 + K_2} \right) \pm \sqrt{\left( 1 - \rho_H \left( \overline{J_4 + K_4} \right)^2 \right) \left( 1 - \rho_H \left( \overline{J_4 - K_4} \right)^2 \right)} \end{aligned}$$

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<sup>37</sup>Because each of  $W_4, X_4, Y_4, Z_4$ , and their primed counterparts, lie between 0 and  $\pi/4$ , we're assured that cosines of pairwise sums and differences are non-negative. With pseudoface quantities  $J_4, K_4$ , and *their* primed counterparts, the bounds are 0 and  $\pi/2$ ; while the cosines of their differences are necessarily non-negative, we lack immediate assurance for the sums. However, the "product" result in the following relations can be verified via more fundamental means—expressing everything in terms of edge lengths—so that the *ratio* of the cosines of the pseudoface differences must be non-negative.



A. STANDARD NOTATION

This note (in the tradition of most notes on hedronometry by this author) reserves much—in fact, just-over half— of the Latin alphabet for specific notational use, as described below. Generally, lower-case letters indicate “trigonometric” elements (sides and planar angles, and their measures), while upper-case letters indicate “hedronometric” elements (faces and dihedral angles, and their measures); the lower- and upper-case counterparts of a letter indicate elements in a natural correspondence.

We avoid the use of integer subscripts as indices; rather, we reserve them for compact representation of fractions in formulas. (See Section A.2.)

**Note.** Without fear of confusion, a symbol represents both a geometric element and its measure. (E.g., edge  $a$  has length  $a$ ; angle  $B$  has measure  $B$ ; face  $Z$  has area  $Z$ .)

**A.1. Tetrahedral Elements** ( $a, b, c, d, e, f; A, B, C, D, E, F; W, X, Y, Z; H, J, K; h, j, k; \tau_h, \tau_j, \tau_k$ ). A tetrahedron’s edges  $a, b, c$  concur at a vertex and oppose respective edges  $d, e, f$ . The dihedral angles along the edges are, respectively,  $A, B, C$  and  $D, E, F$ .

The tetrahedron’s faces are

$$W := \triangle def \quad X := \triangle dbc \quad Y := \triangle aec \quad Z := \triangle abf$$

When discussing “right-corner” tetrahedra, we take edges  $a, b, c$  to be mutually-orthogonal, and angles  $A, B, C$  to be right angles. This makes  $W$  the figure’s “hypotenuse-face” and  $X, Y, Z$  its right-triangular “leg-faces”.

We sometimes (as in Section 5) assign symbols  $w, x, y, z$  to the altitudes to faces  $W, X, Y, Z$ , but other times revert using these symbols for generic quantities and coordinates in the traditional way.

With regard to pseudo-elements<sup>38</sup> (and their measures), we make the following associations between various pairs of traditional elements and trios comprising a pseudoface ( $H, J, K$ ), a pseudo-altitude ( $h, j, k$ ), and a twist angle ( $\tau_h, \tau_j, \tau_k$ ):

$$\begin{aligned} a, d &\leftrightarrow A, D \leftrightarrow (Y, Z), (W, X) \leftrightarrow H, h, \tau_h \\ b, e &\leftrightarrow B, E \leftrightarrow (Z, X), (W, Y) \leftrightarrow J, j, \tau_j \\ c, f &\leftrightarrow C, F \leftrightarrow (X, Y), (W, Z) \leftrightarrow K, k, \tau_k \end{aligned}$$

**A.2. Trigonometry** ( $\ddot{x}, \bar{x}, \hat{x}, \llbracket x, y, z \rrbracket, [x, y, z], [x, y, z]$ ). To remove visual clutter from trigonometric expressions, we adopt this “Morse Code” (and hat) notation:

$$\begin{aligned} \ddot{x} &:= \cosh x & \text{and} & \quad \bar{x} := \sinh x & \text{and} & \quad \hat{x} := \tanh x & , \text{ for lengths } x \\ \ddot{\theta} &:= \cos \theta & \text{and} & \quad \bar{\theta} := \sin \theta & \text{and} & \quad \hat{\theta} := \tan \theta & , \text{ for angles } \theta \\ \ddot{X} &:= \cos X & \text{and} & \quad \bar{X} := \sin X & \text{and} & \quad \hat{X} := \tan X & , \text{ for areas } X \end{aligned}$$

Context should make clear whether hyperbolic or circular functions are intended.

<sup>38</sup>See footnotes 1 and 14 for definitions of *pseudoface* and *pseudo-altitude*.

“Half-measures” (half-angles, half-lengths, half-areas) and even “quarter-measures” appear so often in formulas that writing (and reading) them as fractions can be a strain. For typographic simplicity, we write “ $x_n$ ” for “ $x/n$ ”.

Finally, we define the “Heronic sine and cosine products”<sup>39</sup> via

$$(96a) \quad \begin{aligned} \llbracket x, y, z \rrbracket &:= \overline{(x_2 + y_2 + z_2)} \overline{(-x_2 + y_2 + z_2)} \overline{(x_2 - y_2 + z_2)} \overline{(x_2 + y_2 - z_2)} \\ &= \frac{1}{4} (2\ddot{x}\ddot{y}\ddot{z} - \ddot{x}^2 - \ddot{y}^2 - \ddot{z}^2 + 1) \end{aligned}$$

$$(96b) \quad \begin{aligned} \llbracket x, y, z \rrbracket &:= \overline{\overline{\overline{\overline{(x_2 + y_2 + z_2)} (-x_2 + y_2 + z_2) (x_2 - y_2 + z_2) (x_2 + y_2 - z_2)}}}} \\ &= \frac{1}{4} (2\ddot{x}\ddot{y}\ddot{z} + \ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2 - 1) \end{aligned}$$

These take their name and symbology from the so-called and -symbolized (by me) algebraic Heronic product

$$(97) \quad [x, y, z] := (x_2 + y_2 + z_2) (-x_2 + y_2 + z_2) (x_2 - y_2 + z_2) (x_2 + y_2 - z_2)$$

that gives Heron’s formula for the square of the area of a *Euclidean* triangle with edges  $x, y, z$ . This note occasionally uses (97) to abbreviate the characteristic four-part product.<sup>40</sup>

We’ll note the “side-angle-side” form of the Heronic (hyperbolic) sine product of edges  $x, y, z$  of a hyperbolic triangle, with angles  $\theta, \phi, \psi$  opposite respective sides  $x, y, z$ :

$$(98) \quad \llbracket x, y, z \rrbracket = \frac{1}{4} \overline{\overline{\overline{\overline{y^2 z^2 \theta^2}}} = \frac{1}{4} \overline{\overline{\overline{\overline{z^2 x^2 \phi^2}}} = \frac{1}{4} \overline{\overline{\overline{\overline{x^2 y^2 \psi^2}}}}$$

This equation drives the Law of “Side-Angle-Side-Altitude” Sines (Theorem 5.1).

**A.3. Matrices, Determinants, Conjugation Ratio, and Minors** ( $\widehat{G}, \widehat{g}, M_*, m_*$ ). Define the traditional (“angle”) Gram matrix,  $G$ , and conjugate (“edge”) Gram matrix,  $g$ , with rows and columns indexed by faces  $W, X, Y, Z$  (in order).<sup>41</sup>

$$(99) \quad G := \begin{bmatrix} 1 & -\ddot{D} & -\ddot{E} & -\ddot{F} \\ -\ddot{D} & 1 & -\ddot{C} & -\ddot{B} \\ -\ddot{E} & -\ddot{C} & 1 & -\ddot{A} \\ -\ddot{F} & -\ddot{B} & -\ddot{A} & 1 \end{bmatrix} \quad g := \begin{bmatrix} -1 & -\ddot{a} & -\ddot{b} & -\ddot{c} \\ -\ddot{a} & -1 & -\ddot{f} & -\ddot{e} \\ -\ddot{b} & -\ddot{f} & -1 & -\ddot{d} \\ -\ddot{c} & -\ddot{e} & -\ddot{d} & -1 \end{bmatrix}$$

<sup>39</sup>The equivalence of factored and expanded forms holds for both circular and hyperbolic functions.

<sup>40</sup>See, for instance, (9a) and (41c). Note that, because I incorporate the halving of arguments in building the factors (to provide uniformity with the sine and cosine products), I must *double* arguments — or else multiply the result by 16 — when I simply want to use the “[.]” notation to compress these four-fold products.

<sup>41</sup>The angles in a row (column) of  $G$  *surround* the index-face, while remaining angles *oppose* that face; conjugately, the edges in a row (column) of  $g$  *oppose* the index-face, while the remaining edges *surround* it.

The determinants of these matrices are as follows:

$$(100a) \quad \widehat{G} := \det G = \begin{pmatrix} 1 - \ddot{A}^2 - \ddot{B}^2 - \ddot{C}^2 - \ddot{D}^2 - \ddot{E}^2 - \ddot{F}^2 \\ -2\ddot{A}\ddot{B}\ddot{C} - 2\ddot{A}\ddot{E}\ddot{F} - 2\ddot{D}\ddot{B}\ddot{F} - 2\ddot{D}\ddot{E}\ddot{C} \\ +\ddot{A}^2\ddot{D}^2 + \ddot{B}^2\ddot{E}^2 + \ddot{C}^2\ddot{F}^2 \\ -2\ddot{A}\ddot{D}\ddot{B}\ddot{E} - 2\ddot{B}\ddot{E}\ddot{C}\ddot{F} - 2\ddot{C}\ddot{F}\ddot{A}\ddot{D} \end{pmatrix} \leq 0$$

$$(100b) \quad \widehat{g} := \det g = \begin{pmatrix} 1 - \ddot{a}^2 - \ddot{b}^2 - \ddot{c}^2 - \ddot{d}^2 - \ddot{e}^2 - \ddot{f}^2 \\ +2\ddot{d}\ddot{b}\ddot{c} + 2\ddot{a}\ddot{e}\ddot{c} + 2\ddot{a}\ddot{b}\ddot{f} + 2\ddot{d}\ddot{e}\ddot{f} \\ +\ddot{a}^2\ddot{d}^2 + \ddot{b}^2\ddot{e}^2 + \ddot{c}^2\ddot{f}^2 - 2\ddot{a}\ddot{d}\ddot{b}\ddot{e} - 2\ddot{a}\ddot{d}\ddot{c}\ddot{f} - 2\ddot{b}\ddot{e}\ddot{c}\ddot{f} \end{pmatrix} \leq 0$$

and we define (what I call) the *conjugation ratio*<sup>42</sup> thusly:

$$(101) \quad \Gamma := \frac{\widehat{G}}{\widehat{g}}$$

For face-indices  $P$  and  $Q$ , define the  $(PQ)$ -th cofactors,  $M_{PQ}$  and  $m_{PQ}$ :

$$(102) \quad M_{PQ} := (-1)^{P+Q} \text{minor}_{PQ} G \quad m_{PQ} := (-1)^{P+Q} \text{minor}_{PQ} g$$

Thus,

$$(103a) \quad M_{WX} = \ddot{A} \left( -\ddot{A}\ddot{D} + \ddot{B}\ddot{E} + \ddot{C}\ddot{F} \right) + \ddot{D} + \ddot{B}\ddot{F} + \ddot{C}\ddot{E}$$

$$(103b) \quad m_{WX} = \ddot{d} \left( -\ddot{a}\ddot{d} + \ddot{b}\ddot{e} + \ddot{c}\ddot{f} \right) + \ddot{a} - \ddot{b}\ddot{f} - \ddot{c}\ddot{e}$$

When  $P = Q$ , we use a single index. For instance,

$$(104a) \quad M_W = -2\ddot{A}\ddot{B}\ddot{C} - \ddot{A}^2 - \ddot{B}^2 - \ddot{C}^2 + 1 = -4[A, B, C]$$

$$(104b) \quad m_W = -2\ddot{d}\ddot{e}\ddot{f} + \ddot{d}^2 + \ddot{e}^2 + \ddot{f}^2 - 1 = -4[d, e, f] = -16\overline{W}_2^2 \ddot{d}_2^2 \ddot{e}_2^2 \ddot{f}_2^2$$

We note that Mednykh and Pashkevich [3, Proposition 4] provide cofactor formulas for the lengths of a tetrahedron's edges and altitudes:

$$(105a) \quad \ddot{a}^2 = \frac{M_{WX}^2}{M_W M_X} \quad \ddot{d}^2 = \frac{M_{YZ}^2}{M_Y M_Z} \quad \overline{w}^2 = \frac{-\widehat{G}}{M_W} = \frac{\widehat{g}}{m_W}$$

and we deduce

$$(105b) \quad \overline{a}^2 = \frac{-\overline{A}^2 \widehat{G}}{M_W M_X} \quad \overline{d}^2 = \frac{-\overline{D}^2 \widehat{G}}{M_Y M_Z}$$

$$(105c) \quad \widehat{a} = \frac{\overline{A} \sqrt{-\widehat{G}}}{M_{WX}} \quad \widehat{d} = \frac{\overline{D} \sqrt{-\widehat{G}}}{M_{YZ}}$$

<sup>42</sup>Note the use of upper-case gamma as a reminder that the numerator of the conjugation ratio contains the upper-cased —that is, *angle-based*— determinant,  $\widehat{G}$ . I'm tempted to introduce “ $\gamma$ ” to denote the reciprocal ratio, just for the sake of completeness.

Likewise, we have cofactor formulas for dihedral angles:

$$(105d) \quad \ddot{D}^2 = \frac{m_{WX}^2}{m_W m_X} \quad \ddot{A}^2 = \frac{m_{YZ}^2}{m_Y m_Z}$$

$$(105e) \quad \bar{D}^2 = \frac{-\bar{a}^2 \widehat{G}}{m_W m_X} \quad \bar{A}^2 = \frac{-\bar{d}^2 \widehat{G}}{m_Y m_Z}$$

$$(105f) \quad \widehat{D} = \frac{\bar{a} \sqrt{-\widehat{G}}}{m_{WX}} \quad \widehat{A} = \frac{\bar{d} \sqrt{-\widehat{G}}}{m_{YZ}}$$

Also, we have this formula for a face area in terms of the dihedral angles:

$$(106) \quad \begin{aligned} 4\bar{W}_2^2 \bar{D}^2 \bar{E}^2 \bar{F}^2 &= -\widehat{G} + \bar{D}^2 M_X + \bar{E}^2 M_Y + \bar{F}^2 M_Z \\ &\quad - 2(\ddot{A} + \ddot{E}\ddot{F}) \sqrt{M_Y M_Z} - 2(\ddot{B} + \ddot{F}\ddot{D}) \sqrt{M_Z M_X} \\ &\quad - 2(\ddot{C} + \ddot{D}\ddot{E}) \sqrt{M_X M_Y} \end{aligned}$$

## B. COORDINATES

In coordinatized hyperbolic space,<sup>43</sup> let  $P(x_p, y_p, z_p)$  be such that

- $z_p$  measures the *signed* length of a perpendicular dropped from  $P$  to a point  $P_{xy}$  in the  $xy$ -plane;
- $y_p$  measures the *signed* length of a perpendicular dropped from  $P_{xy}$  to a point  $P_x$  on the  $x$ -axis; and
- $x_p$  measures the *signed* distance from  $P_x$  to the origin.

Every coordinate system has a distance formula, and here's the one for this context:

**Theorem B.1** (The Distance Formula in Coordinatized Hyperbolic Space). *The distance,  $|AB|$ , from point  $A(x_a, y_a, z_a)$  to point  $B(x_b, y_b, z_b)$  is given by*

$$(107) \quad \cosh |AB| = \ddot{z}_a \ddot{z}_b ( \ddot{y}_a \ddot{y}_b ( \ddot{x}_a \ddot{x}_b - \overline{x_a x_b} ) - \overline{y_a y_b} ) - \overline{z_a z_b}$$

We wish to describe a coordinatization of a tetrahedron. Let the tetrahedron have elements with standard<sup>44</sup> edge, (pseudo)face, and pseudo-altitude labels, and let vertices  $P, Q, R, S$  be opposite respective faces  $W, X, Y, Z$ . We align pseudo-altitude  $h$  with the  $x$ -axis and an adjacent edge  $PQ$  with the  $y$ -axis; say that the pseudo-altitude separates  $PQ$  into segments of length  $a_P$  and  $a_Q$ , and  $RS$  into segments of length  $d_R$  and  $d_S$ . We write

$$P(0, -a_P, 0) \quad Q(0, a_Q, 0) \quad R(h, -y_R, -z_R) \quad S(h, y_S, z_S)$$

where

$$\begin{aligned} \tanh y_R &= \tanh d_R \cos \tau_h & \tanh y_S &= \tanh d_S \cos \tau_h \\ \sinh z_R &= \sinh d_R \sin \tau_h & \sinh z_S &= \sinh d_S \sin \tau_h \end{aligned}$$

for  $\tau_h$  the angular *twist* of edge  $RS$  out of the  $xy$ -plane.

This arrangement, and the Distance Formula, provide these formulas for edge lengths.<sup>45</sup>

$$(108) \quad \begin{aligned} \ddot{a} &= \cosh |PQ| = \ddot{a}_P \ddot{a}_Q + \overline{a_P a_Q} & \ddot{d} &= \cosh |RS| = \ddot{d}_R \ddot{d}_S + \overline{d_R d_S} \\ \ddot{b} &= \cosh |PR| = \ddot{a}_P \ddot{d}_R \ddot{h} - \overline{a_P d_R \tau_h} & \ddot{e} &= \cosh |SQ| = \ddot{a}_Q \ddot{d}_S \ddot{h} - \overline{a_Q d_S \tau_h} \\ \ddot{c} &= \cosh |PS| = \ddot{a}_P \ddot{d}_S \ddot{h} + \overline{a_P d_S \tau_h} & \ddot{f} &= \cosh |QR| = \ddot{a}_Q \ddot{d}_R \ddot{h} + \overline{a_Q d_R \tau_h} \end{aligned}$$

With these, the conjugate Gram determinant,  $\hat{g}$ , (see (100b)) collapses from over a dozen terms to just one:

$$(109) \quad \hat{g} = -\ddot{a}^2 \ddot{d}^2 \ddot{h}^2 \tau_h^2$$

<sup>43</sup>Here, we enlist  $x, y, z$  as symbols for standard coordinates instead of tetrahedral altitudes.

<sup>44</sup>See Appendix A.

<sup>45</sup>For instance,

$$\ddot{b} = \ddot{z}_R ( \ddot{a}_P \ddot{y}_R ( \ddot{h} - 0 ) - \overline{a_P y_R} ) - 0 = \ddot{a}_P ( \ddot{y}_R \ddot{z}_R ) \ddot{h} - \overline{a_P ( \ddot{y}_R \ddot{z}_R )} = \ddot{a}_P \ddot{d}_R \ddot{h} - \overline{a_P d_R \tau_h}$$

where the simplifications  $\ddot{y}_R \ddot{z}_R = \ddot{d}_R$  and  $\overline{y_R z_R} = \overline{d_R \tau_h}$  follow from the trigonometry of the right triangle with hypotenuse  $d_R$ , legs  $y_R$  and  $z_R$ , and angle  $\tau_h$ .

whence, all un-squared sines being non-negative while  $\widehat{g}$  being non-positive,

$$(110a) \quad \bar{a} \bar{d} \bar{h} \bar{\tau}_h = \sqrt{-\widehat{g}}$$

Also, one readily verifies this identity,

$$(110b) \quad \bar{a} \bar{d} \bar{h} |\dot{\tau}_h| = \left| \ddot{c}\dot{f} - \ddot{b}\dot{e} \right|$$

where the supplementary ambiguity of  $\tau_h$  necessitates the use of absolute values. We re-confirm a corollary (42) to the formulas for  $h$  and  $\tau_h$ :

$$(110c) \quad \left[ 2\bar{a}\bar{d}, 2 \left( \ddot{b}\dot{e} - \ddot{c}\dot{f} \right), 2i\sqrt{-\widehat{g}} \right] = -\bar{a}^4 \bar{d}^4 \left( \dot{h}^2 - \dot{\tau}_h^2 \right)^2$$

and we offer these Gram matrix cofactor expansions (see (103) and (104)):

$$(110d) \quad m_{WX} = \bar{d}^2 \left( \bar{h}^2 \bar{a}_P \bar{a}_Q - \bar{\tau}_h^2 \bar{a}_P \bar{a}_Q \right)$$

$$(110e) \quad m_W = \bar{d}^2 \left( \bar{\tau}_h^2 - \left( \dot{h}^2 - \dot{\tau}_h^2 \right) \bar{a}_Q^2 \right) = -\bar{d}^2 \left( \bar{h}^2 + \left( \dot{h}^2 - \dot{\tau}_h^2 \right) \bar{a}_Q^2 \right)$$

$$(110f) \quad m_X = \bar{d}^2 \left( \bar{\tau}_h^2 - \left( \dot{h}^2 - \dot{\tau}_h^2 \right) \bar{a}_P^2 \right) = -\bar{d}^2 \left( \bar{h}^2 + \left( \dot{h}^2 - \dot{\tau}_h^2 \right) \bar{a}_P^2 \right)$$

which allow us to express the sub-segment lengths  $a_P$  and  $a_Q$  in terms of full-segment lengths:

$$(111a) \quad \sinh 2a_P = \frac{2\bar{a} (m_{WX} + \ddot{a} m_X)}{\sqrt{- \left[ 2\bar{a}\bar{d}, 2 \left( \ddot{b}\dot{e} - \ddot{c}\dot{f} \right), 2i\sqrt{-\widehat{g}} \right]}}$$

$$(111b) \quad \sinh 2a_Q = \frac{2\bar{a} (m_{WX} + \ddot{a} m_W)}{\sqrt{- \left[ 2\bar{a}\bar{d}, 2 \left( \ddot{b}\dot{e} - \ddot{c}\dot{f} \right), 2i\sqrt{-\widehat{g}} \right]}}$$

so that

$$(111c) \quad \frac{\sinh 2a_P}{\sinh 2a_Q} = \frac{m_{WX} + \ddot{a} m_X}{m_{WX} + \ddot{a} m_W}$$

Solving the above for  $\ddot{a}$ , we find that  $\ddot{a}$  cancels, leaving

$$(111d) \quad m_{WX} \sinh (a_P - a_Q) = -2 (m_W \sinh 2a_P - m_X \sinh 2a_Q)$$

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