

HEDRONOMETRIC FORMULAS FOR A HYPERBOLIC TETRAHEDRON

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ABSTRACT. This “living document” will serve as an ever-expanding resource of results in hyperbolic (tetra)hedronometry. To maximize information density, I minimize discussion and proof (at least in early drafts), but the reader can readily verify the geometry with basic plane trigonometry and the algebra with standard identities; the only truly sophisticated notions are the Schalfli and Derevnin-Mednykh formulas for volume.

(Refer to Appendix A for notation used throughout.)

1. FUNDAMENTALS: THE LAWS OF COSINES

At the core of hedronometry lie three Laws of Cosines that epitomize the field as “dimensionally-enhanced” trigonometry by relating, not the edges and planar angles of a triangle, but the *faces* (W, X, Y, Z) and *dihedral angles* (A, B, C, D, E, F) of a tetrahedron. The Second Law, in particular, also formally defines the figure’s geometrically-amorphous¹ but algebraically-rewarding² *pseudo-faces* (H, J, K).

Theorem 1.1 (The First Law of Cosines).

$$(1) \quad \ddot{W}_2 = \ddot{X}_2\ddot{Y}_2\ddot{Z}_2 - \overline{X_2Y_2Z_2} \sqrt{-4[A, B, C]} + \ddot{X}_2\overline{Y_2Z_2}\ddot{A} + \overline{X_2Y_2}\ddot{Z}_2\ddot{B} + \overline{X_2Y_2}\ddot{Z}_2\ddot{C}$$

Theorem 1.2 (The Second Law of Cosines, and the Definition of $0 \leq H, J, K \leq 2\pi$).

$$(2) \quad \begin{aligned} \ddot{Y}_2\ddot{Z}_2 + \overline{Y_2Z_2}\ddot{A} &= \ddot{H}_2 = \ddot{W}_2\ddot{X}_2 + \overline{W_2X_2}\ddot{D} \\ \ddot{Z}_2\ddot{X}_2 + \overline{Z_2X_2}\ddot{B} &= \ddot{J}_2 = \ddot{W}_2\ddot{Y}_2 + \overline{W_2Y_2}\ddot{E} \\ \ddot{X}_2\ddot{Y}_2 + \overline{X_2Y_2}\ddot{C} &= \ddot{K}_2 = \ddot{W}_2\ddot{Z}_2 + \overline{W_2Z_2}\ddot{F} \end{aligned}$$

Theorem 1.3 (The Third Law of Cosines).

$$(3) \quad \begin{aligned} 0 &= 1 - \ddot{W}_2^2 - \ddot{X}_2^2 - \ddot{Y}_2^2 - \ddot{Z}_2^2 - 4\overline{W_2X_2Y_2Z_2} - \ddot{H}_2^2 - \ddot{J}_2^2 - \ddot{K}_2^2 - 2\ddot{H}_2\ddot{J}_2\ddot{K}_2 \\ &+ 2\ddot{H}_2(\ddot{W}_2\ddot{X}_2 + \ddot{Y}_2\ddot{Z}_2) + 2\ddot{J}_2(\ddot{W}_2\ddot{Y}_2 + \ddot{Z}_2\ddot{X}_2) + 2\ddot{K}_2(\ddot{W}_2\ddot{Z}_2 + \ddot{X}_2\ddot{Y}_2) \end{aligned}$$

¹In Euclidean space, a pseudo-face is a quadrilateral determined by projection of the tetrahedron into a plane parallel to a pair of opposite edges; no geometric interpretation of a general hyperbolic pseudo-face is known. (See Section 6 for some special cases.) Nevertheless, we declare that the element’s “area” satisfies the Second Law of Cosines, and that the area is bounded as if the element were a hyperbolic quadrilateral.

²I doubt that the significance of pseudo-faces in catalyzing tetrahedral analysis can be overstated.

These give rise to symbolic abbreviations³ that we'll find helpful:

$$(4a) \quad \langle W_2 \rangle := -\ddot{W}_2 - 2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2 + \ddot{H}_2\ddot{X}_2 + \ddot{J}_2\ddot{Y}_2 + \ddot{K}_2\ddot{Z}_2 \geq 0$$

$$(4b) \quad \langle H_2 \rangle := \ddot{H}_2 + \ddot{J}_2\ddot{K}_2 - \ddot{W}_2\ddot{X}_2 - \ddot{Y}_2\ddot{Z}_2 \geq 0$$

We also note this consequence of (2)

$$(5) \quad \llbracket H_2, W_2, X_2 \rrbracket = \frac{1}{4} \overline{W_2^2 X_2^2 D^2} \geq 0$$

and this Law-of-Cosines-like equality that follows upon reduction *modulo* (3):

$$(6) \quad \langle J_2 \rangle^2 + \langle K_2 \rangle^2 - 2\ddot{H}_2\langle J_2 \rangle\langle K_2 \rangle = 16 \llbracket H_2, W_2, X_2 \rrbracket \llbracket H_2, Y_2, Z_2 \rrbracket$$

2. FROM LENGTHS TO AREAS, AND BACK AGAIN

Standard formulas give the area of a tetrahedron's face from lengths of bounding edges

$$(7) \quad \ddot{W}_2 = \frac{\ddot{d}_2^2 + \ddot{e}_2^2 + \ddot{f}_2^2 - 1}{2\ddot{d}_2\ddot{e}_2\ddot{f}_2} \quad \overline{W_2} = \frac{\sqrt{\llbracket d, e, f \rrbracket}}{2\ddot{d}_2\ddot{e}_2\ddot{f}_2}$$

and we can determine pseudo-face areas from lengths of edges in conspicuous pairs

$$(8a) \quad \ddot{H}_2 = \frac{-\overline{a_2^2 d_2^2} + \overline{b_2^2 e_2^2} + \overline{c_2^2 f_2^2}}{2 \overline{b_2 e_2} \overline{c_2 f_2}} \quad \overline{H_2} = \frac{\sqrt{\llbracket 2\overline{a_2 d_2}, 2\overline{b_2 e_2}, 2\overline{c_2 f_2} \rrbracket}}{2 \overline{b_2 e_2} \overline{c_2 f_2}}$$

$$(8b) \quad \ddot{H}_4^2 = \frac{-\overline{a_2^2 d_2^2} + (\overline{b_2 e_2} + \overline{c_2 f_2})^2}{4 \overline{b_2 e_2} \overline{c_2 f_2}} \quad \overline{H_4}^2 = \frac{\overline{a_2^2 d_2^2} - (\overline{b_2 e_2} - \overline{c_2 f_2})^2}{4 \overline{b_2 e_2} \overline{c_2 f_2}}$$

From these, we can verify

$$(9) \quad \begin{aligned} \langle W_2 \rangle \overline{a_2 b_2 c_2} &= \langle X_2 \rangle \overline{a_2 e_2 f_2} = \langle Y_2 \rangle \overline{d_2 b_2 f_2} = \langle Z_2 \rangle \overline{d_2 e_2 c_2} \\ &= \langle H_2 \rangle \overline{a_2 d_2} = \langle J_2 \rangle \overline{b_2 e_2} = \langle K_2 \rangle \overline{c_2 f_2} = \frac{-\widehat{g}}{64\overline{a_2 b_2 c_2 d_2 e_2 f_2}} \end{aligned}$$

Consequently, we have hedronometric formulas⁴ for edge lengths:

$$(10a) \quad \overline{a_2^2} = \frac{\langle J_2 \rangle \langle K_2 \rangle}{\langle W_2 \rangle \langle X_2 \rangle} \quad \overline{a_2^2} = \frac{4\langle H_2 \rangle \llbracket H_2 Y_2 Z_2 \rrbracket}{\langle W_2 \rangle \langle X_2 \rangle} \quad \widehat{a_2^2} = \frac{4\langle H_2 \rangle \llbracket H_2 Y_2 Z_2 \rrbracket}{\langle J_2 \rangle \langle K_2 \rangle}$$

$$(10b) \quad \overline{d_2^2} = \frac{\langle J_2 \rangle \langle K_2 \rangle}{\langle Y_2 \rangle \langle Z_2 \rangle} \quad \overline{d_2^2} = \frac{4\langle H_2 \rangle \llbracket H_2 W_2 X_2 \rrbracket}{\langle Y_2 \rangle \langle Z_2 \rangle} \quad \widehat{d_2^2} = \frac{4\langle H_2 \rangle \llbracket H_2 W_2 X_2 \rrbracket}{\langle J_2 \rangle \langle K_2 \rangle}$$

³Non-negativity follows from (9).

⁴The sine formulas derive from the cosine formulas, reduced *modulo* the Third Law of Cosines (3).

and, in turn, a symmetric equality involving opposing pairs of edges⁵

$$(11) \quad \frac{\overline{a^2 d^2}}{\llbracket H_2 Y_2 Z_2 \rrbracket \llbracket H_2 W_2 X_2 \rrbracket} = \frac{\overline{b^2 e^2}}{\llbracket J_2 Z_2 X_2 \rrbracket \llbracket J_2 W_2 Y_2 \rrbracket} = \frac{\overline{c^2 f^2}}{\llbracket K_2 X_2 Y_2 \rrbracket \llbracket K_2 W_2 Z_2 \rrbracket}$$

$$= \left(\frac{16 \langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle}{\langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle} \right)^2$$

3. GRAM RELATIONS

Consider the face-indexed (“angle”) Gram matrix, G , and conjugate (“edge”) Gram matrix, g , detailed in Appendix A.3. Their determinants —and the *conjugation ratio* Γ — have the following hedronometric forms:

$$(12a) \quad \widehat{G} := \det G = \frac{-4 \langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle}{\overline{W_2^2 X_2^2 Y_2^2 Z_2^2}} \leq 0 \quad \widehat{g} := \det g = \frac{-4^3 \langle H_2 \rangle^3 \langle J_2 \rangle^3 \langle K_2 \rangle^3}{\langle W_2 \rangle^2 \langle X_2 \rangle^2 \langle Y_2 \rangle^2 \langle Z_2 \rangle^2} \leq 0$$

$$(12b) \quad \Gamma := \frac{\widehat{G}}{\widehat{g}} = \left(\frac{\langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle}{4 \langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle \overline{W_2 X_2 Y_2 Z_2}} \right)^2$$

Cofactors of entries along the diagonals of the matrices have these forms:

$$(13a) \quad M_W = 4[A, B, C] = \frac{\langle W_2 \rangle^2}{\overline{X_2^2 Y_2^2 Z_2^2}}$$

$$(13b) \quad m_W = -4[d, e, f] = -16 \overline{W_2^2 d^2 e^2 f^2} = - \left(\overline{W_2} \frac{4 \langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle}{\langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle} \right)^2$$

while, via (70b), the remaining cofactors of G and g have these forms:

$$(14a) \quad M_{WX} = \ddot{a} \sqrt{M_W M_X} = (\ddot{a}^2 + \overline{a_2^2}) \sqrt{M_W M_X} = \frac{\langle J_2 \rangle \langle K_2 \rangle + 4 \langle H_2 \rangle \llbracket H_2 Y_2 Z_2 \rrbracket}{\overline{W_2 X_2 Y_2^2 Z_2^2}}$$

$$(14b) \quad m_{WX} = \ddot{D} \sqrt{m_W m_X} = (\ddot{H}_2 - \ddot{W}_2 \ddot{X}_2) \frac{4^2 \langle H_2 \rangle^2 \langle J_2 \rangle^2 \langle K_2 \rangle^2}{\langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle^2 \langle Z_2 \rangle^2}$$

One can readily verify identities^{6,7} presented by Mednykh and Paskevich in [3], such as:

⁵A cleaner version of this result appears in (15b).

⁶(15a) This follows from our expressions for Γ , M_W , m_W . M&P’s equation (2) —prior to Lemma 1— asserts equality between the ratios and *positive* Γ ; M&P’s derivation includes factors such as (in our notation) $\sqrt{m_W}$, but, by (69b), $m_W \leq 0$. Someone among us has a sign error. (15b) M&P’s Theorem 2, which follows from (5) and (11). (15c) M&P’s Theorem 4. (15d) M&P’s Corollary 1, which follows from (15b) and (15c). The “ \pm_i ”s match for a given i , but are otherwise independent.

⁷M&P’s result relating altitude lengths to Γ appears to be in error; see the footnote for equation (19).

$$(15a) \quad \frac{M_W}{m_W} = \frac{M_X}{m_X} = \frac{M_Y}{m_Y} = \frac{M_Z}{m_Z} = -\Gamma$$

$$(15b) \quad \frac{\overline{AD}}{\overline{ad}} = \frac{\overline{BE}}{\overline{be}} = \frac{\overline{CF}}{\overline{cf}} = \sqrt{\Gamma}$$

$$(15c) \quad \frac{\overline{\overline{AD}} - \overline{\overline{BE}}}{\overline{\overline{ad}} - \overline{\overline{be}}} = \frac{\overline{\overline{BE}} - \overline{\overline{CF}}}{\overline{\overline{be}} - \overline{\overline{cf}}} = \frac{\overline{\overline{CF}} - \overline{\overline{AD}}}{\overline{\overline{cf}} - \overline{\overline{ad}}} = -\sqrt{\Gamma}$$

$$(15d) \quad \frac{\overline{\overline{(A \pm_1 D)}} - \overline{\overline{(B \pm_2 E)}}}{\overline{\overline{(a \pm_1 d)}} - \overline{\overline{(b \pm_2 e)}}} = \frac{\overline{\overline{(B \pm_3 E)}} - \overline{\overline{(C \pm_4 F)}}}{\overline{\overline{(b \pm_3 e)}} - \overline{\overline{(c \pm_4 f)}}} = \frac{\overline{\overline{(C \pm_5 F)}} - \overline{\overline{(A \pm_6 D)}}}{\overline{\overline{(a \pm_5 d)}} - \overline{\overline{(b \pm_6 e)}}} = -\sqrt{\Gamma}$$

We can often (as in Section 4) use the conjugation ratio to catalyze the conversion between equations in edge-related lower-case symbols (a, m_*, g , etc.) and ones in angle- or area-related upper-case symbols (A, M_*, G , etc.), effectively doubling the utility of each result. We call these *conjugate* relations.

For example, write $P := M_W M_X M_Y M_Z$ and $p := m_W m_X m_Y m_Z$. Then

$$(16) \quad P = \left(\frac{\langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle}{\overline{W_2^3 X_2^3 Y_2^3 Z_2^3}} \right)^2 \quad p = \left(\frac{4^4 \langle H_2 \rangle^4 \langle J_2 \rangle^4 \langle K_2 \rangle^4 \overline{W_2 X_2 Y_2 Z_2}}{\langle W_2 \rangle^3 \langle X_2 \rangle^3 \langle Y_2 \rangle^3 \langle Z_2 \rangle^3} \right)^2$$

and, again, one can readily verify these conjugate identities⁸ from [3, Proposition 5]:

$$(17) \quad \widehat{g}P = \widehat{G}^3 \leftrightarrow \widehat{G}p = \widehat{g}^3 \quad P^3 p = \widehat{G}^8 \leftrightarrow p^3 P = \widehat{g}^8$$

4. ALTITUDES AND PSEUDO-ALTITUDES

Let w, x, y, z be altitudes to respective faces W, X, Y, Z . We have conjugate relations⁹

$$(18a) \quad \overline{w}^2 M_W = \overline{x}^2 M_X = \overline{y}^2 M_Y = \overline{z}^2 M_Z = -\widehat{G}$$

$$(18b) \quad \overline{w}^2 m_W = \overline{x}^2 m_X = \overline{y}^2 m_Y = \overline{z}^2 m_Z = \widehat{g}$$

⁸Given the interplay of factors $S := \overline{W_2 X_2 Y_2 Z_2}$, $T := \langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle$, $U := 4 \langle H_2 \rangle \langle J_2 \rangle \langle K_2 \rangle$, one can show that any vanishing product of integer powers of $G = -U/S^2$, $g = -U^3/T^2$, $P = T^2/S^6$, $p = S^2 U^8/T^6$, can be written $(\widehat{G}p/\widehat{g}^3)^\alpha (Pp^3/\widehat{g}^8)^\beta = 1$ for integers α, β . Abusing terminology, the relations $\widehat{G}p = \widehat{g}^3$ and $Pp^3 = \widehat{g}^8$ have all possible relations of this kind in their *integer span*. Likewise, their conjugates. For one relation and its conjugate to span all relations in this way, however, requires non-integer exponents.

⁹Equation (18b) agrees with Mednykh and Pashkevich [3, Proposition 4(ii)], which appears in this note as equation (70b). The second set of equations arises from dividing the first set through by $-\Gamma$. See (15a).

whence¹⁰

$$(19) \quad \overline{wxyz} = \frac{\widehat{G}^2}{\sqrt{M_W M_X M_Y M_Z}} = \frac{\widehat{g}^2}{\sqrt{m_W m_X m_Y m_Z}} = \frac{4^2 \langle H_2 \rangle^2 \langle J_2 \rangle^2 \langle K_2 \rangle^2}{W_2 X_2 Y_2 Z_2 \langle W_2 \rangle \langle X_2 \rangle \langle Y_2 \rangle \langle Z_2 \rangle}$$

By (14a) and (10), we can write

$$(20) \quad \frac{\overline{w^2 x^2} M_{WX}^2}{\overline{a^2}} = \frac{\overline{w^2 y^2} M_{WX}^2}{\overline{b^2}} = \frac{\overline{w^2 z^2} M_{WX}^2}{\overline{c^2}} = \frac{\overline{y^2 z^2} M_{WX}^2}{\overline{d^2}} = \frac{\overline{z^2 x^2} M_{WX}^2}{\overline{e^2}} = \frac{\overline{x^2 y^2} M_{WX}^2}{\overline{f^2}}$$

Also,

$$(21) \quad \overline{w^2 W_2^2} \langle W_2 \rangle^2 = \frac{\widehat{g} \overline{W_2^2} \langle W_2 \rangle^2}{m_W} = \frac{\widehat{g} \langle W_2 \rangle^2 \langle X_2 \rangle^2 \langle Y_2 \rangle^2 \langle Z_2 \rangle^2}{4^2 \langle H_2 \rangle^2 \langle J_2 \rangle^2 \langle K_2 \rangle^2}$$

$$(22) \quad \overline{w^2 W_2^2} \langle W_2 \rangle^2 = \overline{w^2 W_2^2} \overline{X_2^2} \overline{Y_2^2} \overline{Z_2^2} (-4[A, B, C])$$

Let h, j, k be pseudo-altitudes¹¹ between respective edge pairs (a, d) , (b, e) , (c, f) ; and let τ_h, τ_j, τ_k be the corresponding *twists* about those pseudo-altitudes. Equation (74a) from Appendix B gives these conjugate relations

$$(23a) \quad \overline{adh\tau_h} = \overline{bej\tau_j} = \overline{cfk\tau_k} = \sqrt{-\widehat{g}}$$

$$(23b) \quad \overline{ADh\tau_h} = \overline{BEj\tau_j} = \overline{CFk\tau_k} = \sqrt{-\widehat{G}}$$

Recall now from (69b) that m_\star is a Heronic sine product; invoking the product's “side-angle-side” form (63), we see that equalities (18b) and (23a) have an identical structure, giving rise to a result that neatly accommodates both *standard* and *pseudo* elements.^{12,13}

Theorem 4.1 (Law of “Side-Angle-Side-Altitude” Sines). *The product*

$$(24) \quad \sinh(\text{edge}) \cdot \sinh(\text{edge}) \cdot \sin(\text{angle}) \cdot \sinh(\text{altitude})$$

is a metric invariant for hyperbolic tetrahedra, with value $\sqrt{-\widehat{g}}$ for any two distinct “edge”s and the “angle” and “altitude” they determine.¹⁴

¹⁰This *disagrees* with Mednykh and Paskevich [3, Proposition 5(iv)], which claims that the product is $\widehat{G}^2/M_W/M_X/M_Y/M_Z = 1/\Gamma$.

¹¹A pair of skew lines in hyperbolic space admits an orthogonal transversal. (See [2].) A *pseudo-altitude* is such a transversal, joining lines through opposing edges of a tetrahedron. The *twist* is the (supplementarily ambiguous) angle between an edge and the plane of the opposing edge and their mutual pseudo-altitude.

¹²Mednykh and Pashkevich [3, Proposition 6] invoke the area-based form of the Heronic sine product (69b) to arrive at this alternative result

$$\sin(\text{half-face}) \cdot \cosh(\text{half-edge}) \cosh(\text{half-edge}) \cosh(\text{half-edge}) \cdot \sinh(\text{altitude}) = \sqrt{-\widehat{g}}/4$$

for any (standard) “face”, the “edge”s surrounding it, and the “altitude” to it. No *pseudo* elements appear.

¹³The analogous product for Euclidean tetrahedra is “edge · edge · sin(angle) · altitude = 6 · volume”, which is valid for both standard and pseudo elements.

¹⁴For adjacent edges, the “angle” lies between them, and the “altitude” drops to the face containing them; for opposite edges, the “altitude” is the pseudo-altitude joining them, and the “angle” is the corresponding twist. Note that taking the sine of a twist renders the supplementary ambiguity of the twist moot.

Moving to cosines, we expand Appendix B's equation (74b) into conjugate relations

$$(25a) \quad \left| \check{c}\check{f} - \check{b}\check{e} \right| = \overline{a\check{d}}\check{h} |\check{\tau}_h| \quad \left| \check{a}\check{d} - \check{c}\check{f} \right| = \overline{b\check{e}}\check{j} |\check{\tau}_j| \quad \left| \check{b}\check{e} - \check{a}\check{d} \right| = \overline{c\check{f}}\check{k} |\check{\tau}_k|$$

$$(25b) \quad \left| \check{C}\check{F} - \check{B}\check{E} \right| = \overline{A\check{D}}\check{h} |\check{\tau}_h| \quad \left| \check{A}\check{D} - \check{C}\check{F} \right| = \overline{B\check{E}}\check{j} |\check{\tau}_j| \quad \left| \check{B}\check{E} - \check{A}\check{D} \right| = \overline{C\check{F}}\check{k} |\check{\tau}_k|$$

Isolating either of \check{h} or $\check{\tau}_h$ in (23a) and (25a) gives the same result; that is, we find that $u = \check{h}^2$ and $u = \check{\tau}_h^2$ are the two roots in each of these conjugate quadratic equations:

$$(26a) \quad u^2 \overline{a^2 d^2} - u \left(\overline{a^2 d^2} + (\check{c}\check{f} - \check{b}\check{e})^2 - \widehat{g} \right) + (\check{c}\check{f} - \check{b}\check{e})^2 = 0$$

$$(26b) \quad u^2 \overline{A^2 D^2} - u \left(\overline{A^2 D^2} + (\check{C}\check{F} - \check{B}\check{E})^2 - \widehat{G} \right) + (\check{C}\check{F} - \check{B}\check{E})^2 = 0$$

As $\check{h}^2 \geq 1 \geq \check{\tau}_h^2$ for hyperbolic cosine \check{h} and circular cosine $\check{\tau}_h$, we can write specifically

$$(27a) \quad \check{h}^2 = \frac{\overline{a^2 d^2} + (\check{c}\check{f} - \check{b}\check{e})^2 - \widehat{g} + \delta}{2\overline{a^2 d^2}} = \frac{\overline{A^2 D^2} + (\check{C}\check{F} - \check{B}\check{E})^2 - \widehat{G} + \Delta}{2\overline{A^2 D^2}}$$

$$(27b) \quad \check{\tau}_h^2 = \frac{\overline{a^2 d^2} + (\check{c}\check{f} - \check{b}\check{e})^2 - \widehat{g} - \delta}{2\overline{a^2 d^2}} = \frac{\overline{A^2 D^2} + (\check{C}\check{F} - \check{B}\check{E})^2 - \widehat{G} - \Delta}{2\overline{A^2 D^2}}$$

for non-negative discriminants, δ and Δ , which a convenient imaginary unit allows us to express as Heronoic products:

$$(27c) \quad \delta^2 = - \left[2\overline{a\check{d}}, 2(\check{c}\check{f} - \check{b}\check{e}), 2i\sqrt{-\widehat{g}} \right] \quad \Delta^2 = - \left[2\overline{A\check{D}}, 2(\check{C}\check{F} - \check{B}\check{E}), 2i\sqrt{-\widehat{G}} \right]$$

Note that the above implies

$$(28) \quad \check{h}^2 - \check{\tau}_h^2 = \frac{\sqrt{- \left[2\overline{a\check{d}}, 2(\check{b}\check{e} - \check{c}\check{f}), 2i\sqrt{-\widehat{g}} \right]}}{\overline{a^2 d^2}} = \frac{\sqrt{- \left[2\overline{A\check{D}}, 2(\check{B}\check{E} - \check{C}\check{F}), 2i\sqrt{-\widehat{G}} \right]}}{\overline{A^2 D^2}}$$

5. ORTHOGONAL TWISTS AND PERFECT TETRAHEDRA

A right-angle twist about a pseudo-altitude is naturally called *orthogonal*. A tetrahedron is called *perfect*¹⁵ when all three pairs of opposing edges are orthogonally twisted about their pseudo-altitudes. For example, a *right-corner* tetrahedron ($A = B = C = \pi/2$) is orthogonal, as is a *regular* tetrahedron whose faces are equilateral triangles.

By (25a) and (25b), a perfect tetrahedron's opposite elements combine symmetrically:

$$(29) \quad \check{a}\check{d} = \check{b}\check{e} = \check{c}\check{f} \quad \check{A}\check{D} = \check{B}\check{E} = \check{C}\check{F}$$

Using the Second Law of Cosines (2) to re-write the dihedral cosines above, we derive this hedronometric characterization of perfection:

$$(30) \quad \check{H}_2 \langle H_2 \rangle = \check{J}_2 \langle J_2 \rangle = \check{K}_2 \langle K_2 \rangle$$

¹⁵Also *orthocentric* or *orthogonal*. *Perfect* is my own term, coined decades ago, so I'll keep it.

An orthogonal twist simplifies the quadratics in (26), providing concise expression of the length of the corresponding pseudo-altitude; for example,

$$(31) \quad \tau_h = \frac{\pi}{2} \quad \Longrightarrow \quad \bar{h} = \frac{\sqrt{-\hat{g}}}{\bar{a}\bar{d}} = \frac{\sqrt{-\hat{G}}}{\bar{A}\bar{D}}$$

The pseudo-altitudes of a perfect tetrahedron, then, have this symmetric property

$$(32) \quad \bar{h}^2 \bar{j}^2 \bar{k}^2 = \frac{-\hat{g}^3}{\bar{a}^2 \bar{b}^2 \bar{c}^2 \bar{d}^2 \bar{e}^2 \bar{f}^2} = \frac{-\hat{G}^3}{\bar{A}^2 \bar{B}^2 \bar{C}^2 \bar{D}^2 \bar{E}^2 \bar{F}^2} \\ = \frac{\langle H_2 \rangle^3 \langle J_2 \rangle^3 \langle K_2 \rangle^3}{\llbracket H_2 W_2 X_2 \rrbracket \llbracket H_2 Y_2 Z_2 \rrbracket \llbracket J_2 W_2 Y_2 \rrbracket \llbracket J_2 Z_2 X_2 \rrbracket \llbracket K_2 W_2 Z_2 \rrbracket \llbracket K_2 X_2 Y_2 \rrbracket}$$

The dependencies of perfection reduce the degrees of freedom in a tetrahedron's elements from six to four. That is to say, up to symmetry, *a perfect tetrahedron is uniquely determined by the areas of its faces*. In theory, we can remove all references to pseudo-faces in our formulas by solving for H, J, K in terms of W, X, Y, Z via the system of equations comprising (30) and (3); in practice, the results seem symbolically intractable, as with this degree-12 polynomial equation for \ddot{H}_2 :

$$(33a) \quad 0 = 16\ddot{H}_2^{12} - 64\ddot{H}_2^{11} + 24\ddot{H}_2^{10} (4s_X^2 - 3) - 8\ddot{H}_2^9 (8s_X^3 - 7s_Y s_Z - 42s_X) \\ + \dots + s (s + s_Y^2) (s + s_Z^2) (s + s_Y^2 + s_Z^2)$$

where

$$(33b) \quad s := 1 - \ddot{W}_2^2 - \ddot{X}_2^2 - \ddot{Y}_2^2 - \ddot{Z}_2^2 - 4\ddot{W}_2 \ddot{Z}_2 \ddot{Y}_2 \ddot{Z}_2 \\ s_X := \ddot{W}_2 \ddot{X}_2 + \ddot{Y}_2 \ddot{Z}_2 \quad s_Y := \ddot{W}_2 \ddot{Y}_2 + \ddot{Z}_2 \ddot{X}_2 \quad s_Z := \ddot{W}_2 \ddot{Z}_2 + \ddot{X}_2 \ddot{Y}_2$$

This equation and its counterparts may be helpful numerically, but what their sprawling symbolic structure is trying to say about the tetrahedron remains a mystery.¹⁶

6. PSEUDO-FACES: SHADOWS AND THE SEARCH FOR SUBSTANCE

For a Euclidean tetrahedron, the pseudo-face associated with pair of opposite edges is a shadow: the quadrilateral projection of the remaining edges into any plane parallel to that pair. Projection and parallelism are tricky concepts in hyperbolic space, so the appropriate analog of a pseudo-face shadow is non-obvious. We consider here a seemingly-promising construction.

Take a tetrahedron having vertices P, Q, R, S , with $a := |PQ|$ and $d := |RS|$. Suppose the pseudo-altitude h meets PQ and RS at M and N , respectively, and define $p := |MP|$, $q := |MQ|$, $r := |NR|$, $s := |NS|$. Let O be the midpoint of the pseudo-altitude.

¹⁶Much the same is true in Euclidean space, which has its own expansive polynomials for pseudo-face areas, but at least perfect Euclidean tetrahedra admit a quartic formula for volume in terms of face areas.

Emulating the notion of a shadowy projection into a plane parallel to a and d , let P' , Q' , R' , S' be the feet of perpendiculars dropped from respective points P , Q , R , S into the plane perpendicular to h at O . Write $p' := |OP'|$, $q' := |OQ'|$, $r' := |OR'|$, $s' := |OS'|$.

We consider the candidacy of $\square P'R'Q'S'$ for the geometric realization of pseudo-face H . For simplicity, we will assume that edges a and d are orthogonally twisted about pseudo-altitude h , and that $p = r$ (so that also $p' = r'$).

Observe that $\square PMOP'$ is a Lambert quadrilateral with legs of length h_2 and p' , and with a co-leg of length p . A little trigonometry gives

$$\ddot{p}'^2 = \frac{\ddot{p}^2 \ddot{h}_2^2}{1 + \ddot{p}^2 \ddot{h}_2^2} \quad \overline{p}'^2 = \frac{\overline{p}^2}{1 + \ddot{p}^2 \ddot{h}_2^2}$$

Moreover, $\triangle P'OR'$ is an isosceles right triangle with leg length p' and area given by

$$\sin(\triangle P'OR') = \frac{\overline{p}'\overline{p}'}{1 + \ddot{p}'\ddot{p}'} = \frac{\overline{p}^2}{1 + \ddot{p}^2 \ddot{h}}$$

We consider three cases.

- **M and N are midpoints of PQ and RS .** This tetrahedron is symmetric about pseudo-altitude h , and we have that $p = q = r = s = a_2$ and $\ddot{b} = \ddot{c} = \ddot{e} = \ddot{f} = \ddot{p}^2 \ddot{h} = \ddot{a}_2 \ddot{h}$, and that $\square P'R'Q'S'$ comprises four identical copies of $\triangle P'OR'$. Consequently, $\sin(\frac{1}{4}\square P'R'Q'S') = \sin(\triangle P'OR') = \overline{a}_2^2/2/\overline{b}_2^2 = \overline{H}_4$ by (8b). Our "mid-plane shadow" is a geometric realization of pseudo-face H !
- **$M = Q$ and $N = S$.** This tetrahedron is an *orthoscheme*, with $q = s = 0$ and $p = a = d = r$; assign $b := |PR|$, $c := |PS|$, $e := |QS| = h$, $f := |QR|$, so that $\ddot{b} = \ddot{a}^2 \ddot{h}$ and $\ddot{c} = \ddot{f} = \ddot{a} \ddot{h}$. The points Q' and S' coincide at O , so that $\triangle P'OR'$ is $\square P'R'Q'S'$, with area given by $\sin(\triangle P'OR') = \overline{a}^2/2/\overline{b}_2^2$. On the other hand, as the reader can verify, \overline{H} via (8) has an incompatible, far-less-concise expression, so that this "mid-plane shadow" is *not* a realization of pseudo-face H .
- **$h = 0$.** This tetrahedron is *degenerate*, with coplanar vertices. We can abandon the orthogonal twist condition, and the supposition $p = r$, but let us assume that edges a and d themselves—not merely the lines determined by them—intersect as diagonals of *convex* quadrilateral $\square PRQS$. (Non-intersecting circumstances are handled similarly.) Then the quadrilateral is the union of the tetrahedron's faces in two ways: $|\square PRQS| = Y + Z = W + X$. But also, $A = D = \pi$, so that $\ddot{H}_2 = \cos(Y_2 + Z_2) = \cos(W_2 + X_2)$. The quadrilateral—arguably, the shadow of the degenerate tetrahedron into a plane "perpendicular" to pseudo-altitude h —realizes pseudo-face H . Moreover, the *bow-tie* quadrilateral $\square PSRQ$, with diagonals (say) b and e , has area¹⁷ $|\square PSRQ| = |Z - X| = |W - Y|$, whereas $B = E = 0$ and $\ddot{J}_2 = \cos(Z_2 - X_2) = \cos(W_2 - Y_2)$; the bow-tie realizes pseudo-face J . Bow-tie

¹⁷We take a bow-tie's area as the absolute value of the difference of its triangular regions, as is consistent with tracing the four edges of the quadrilateral in order, traversing the triangles in different orientations.

$\square PRSQ$, likewise, realizes K . Note that, while the bow-ties may be considered *shadows* of the tetrahedron, and in a plane (degenerately) “parallel” to pairs of opposing edges, they are *not* in a plane perpendicular to the pseudo-altitudes corresponding to J and K .

The symmetric case demonstrates that a straightforward construction can realize a pseudo-face as a “shadow”, and the degenerate case further correlates pseudo-faces with shadows; the orthoscheme case, however, indicates that there is work yet to do in devising a general construction of pseudo-face shadows, if indeed pseudo-faces *are* shadows in general.

7. SPECIAL TETRAHEDRA

7.1. Regular Tetrahedra, $A = B = C = D = E = F$. If the dihedral angles of a hyperbolic tetrahedron match (so that the figure is necessarily perfect, by (29)), then its side lengths match ($a = b = c = d = e = f$), its face areas match ($W = X = Y = Z$), and its pseudo-face areas match ($H = J = K$). Moreover, any one metric —angle measure A , side length a , face area X , pseudo-face area H — completely determines the tetrahedron. A sampling of relations among these metrics follows.

The most straightforward route to relating A and X passes through the realms of hyperbolic and spherical trigonometry. We recall that, by the definition of hyperbolic area, the plane angle θ at any vertex of our equilateral faces satisfies $X = \pi - 3\theta$, so that $\theta = \pi_3 - X_3$. The dihedral angle A relates to face angle θ by the spherical Law of Cosines for Sides:

$$(34) \quad \ddot{A} = \frac{\ddot{\theta} - \ddot{\theta}\ddot{\theta}}{\ddot{\theta}\ddot{\theta}} = \frac{\ddot{\theta}}{1 + \ddot{\theta}} = \frac{\cos(\pi_3 - X_3)}{1 + \cos(\pi_3 - X_3)} = \frac{\ddot{U}_3}{1 + \ddot{U}_3}$$

where $U := \pi - X$. By the Second Law of Cosines (2),

$$(35) \quad \ddot{H}_2 = \ddot{X}_2^2 + \overline{X}_2^2 \ddot{A} = \frac{1}{2} (1 + 4\ddot{U}_3 - 4\ddot{U}_3^2) \implies \overline{H}_4 = \frac{1}{2} (2\ddot{U}_3 - 1)$$

We can also derive

$$(36a) \quad \overline{X}_2^2 = \overline{H}_4^2 (3 + 2\overline{H}_4) \quad \ddot{X}_2^2 = (1 + \overline{H}_4)^2 (1 - 2\overline{H}_4)$$

$$(36b) \quad \langle X_2 \rangle = 4\overline{H}_4^3 (1 + \overline{H}_4) / \sqrt{1 - 2\overline{H}_4} \quad \langle H_2 \rangle = 4\overline{H}_4^3 (1 + \overline{H}_4)$$

$$(36c) \quad \llbracket H_2, X_2, X_2 \rrbracket = 2\overline{H}_4^4 (1 + \overline{H}_4)$$

$$(36d) \quad \ddot{a}_2^2 = \frac{1}{1 - 2\overline{H}_4} \quad \overline{a}_2^2 = \frac{2\overline{H}_4}{1 - 2\overline{H}_4} \quad \widehat{a}_2^2 = 2\overline{H}_4 = 2\ddot{U}_3 - 1 = \frac{3\ddot{A} - 1}{1 - \ddot{A}}$$

7.2. Isohedral Right-Corner Tetrahedra, $A = B = C = \pi/2, D = E = F$. Here, three areas match ($X = Y = Z$), all pseudo-face areas match ($H = J = K$), and triples of side lengths match ($a = b = c$ and $d = e = f$, with $\ddot{d} = \ddot{a}^2$). The First Law of Cosines relates hypotenuse-face W and leg-face X

$$(37) \quad \ddot{W}_2 = \ddot{X}_2^3 - \overline{X}_2^3 = \frac{1}{2} (2 + \overline{X}) \sqrt{1 - \overline{X}} \quad \overline{W}_2^2 = \frac{1}{4} \overline{X}^2 (3 + \overline{X})$$

but the spherical Law of Cosines applied to plane angle $\theta := \pi_3 - W_3$ in hypotenuse W and (acute) plane angle $\phi := \pi_4 - X_2$ in leg X gives something more direct:

$$(38) \quad 0 = \ddot{A} = \frac{\ddot{\theta} - \ddot{\phi}\ddot{\phi}}{\overline{\phi\phi}} \implies \ddot{\phi}^2 = \ddot{\theta} \implies \overline{X} = 2(\overline{\pi_3 - W_3}) - 1$$

From the Second Law of Cosines (2),

$$(39) \quad \ddot{X}_2^2 = \ddot{H}_2 = \ddot{W}_2\ddot{X}_2 + \overline{W_2X_2}\ddot{D}$$

so that

$$(40) \quad \ddot{D} = \frac{\ddot{T}_2}{\sqrt{1 + \ddot{T}_2^2}} = \frac{\sqrt{2\ddot{U}_3}}{2\ddot{U}_6} \quad \overline{D} = \frac{1}{\sqrt{1 + \ddot{T}_2^2}} = \frac{\sqrt{2}}{2\ddot{U}_6}$$

where $T := \pi_2 - X$ and $U := \pi - W$. Also,

$$(41a) \quad \langle H_2 \rangle = \ddot{X}_2\overline{X_2}^3 \quad \langle W_2 \rangle = \overline{X_2}^3 \quad \langle X_2 \rangle = \ddot{X}_2\overline{X_2}^3(\ddot{X}_2 - \overline{X_2})$$

$$(41b) \quad \llbracket H_2X_2X_2 \rrbracket = \frac{1}{4}\overline{X_2}^4 \quad \llbracket H_2W_2X_2 \rrbracket = \frac{1}{2}\ddot{X}_2^2\overline{X_2}^4$$

$$(41c) \quad \ddot{a}_2^2 = \frac{\ddot{X}_2}{\ddot{X}_2 - \overline{X_2}} \quad \overline{a}_2^2 = \frac{\overline{X_2}}{\ddot{X}_2 - \overline{X_2}} \quad \widehat{a}_2^2 = \widehat{X_2}$$

$$(41d) \quad \ddot{d}_2^2 = \frac{1}{1 - \overline{X}} \quad \overline{d}_2^2 = \frac{\overline{X}}{1 - \overline{X}} \quad \widehat{d}_2^2 = \overline{X} = 2\ddot{U}_3 - 1 = \frac{3\ddot{D}^2 - 1}{\overline{D}^2}$$

8. VOLUME

Volume seems the least-scrutable measurement in hyperbolic space, even for tetrahedra. The principle formula is a differential equation from Schlafli [4] in 1950:

$$(42) \quad dV = -a_2 dA - b_2 dB - c_2 dC - d_2 dD - e_2 dE - f_2 dF$$

Derevnin and Mednykh's 2005 solution [1] takes the form of a monolithic integral:

$$(43) \quad V = -\frac{1}{4} \int_{\alpha-\beta}^{\alpha+\beta} \log \frac{\cos \frac{A+B+C+\theta}{2} \cos \frac{A+E+F+\theta}{2} \cos \frac{D+B+F+\theta}{2} \cos \frac{D+E+C+\theta}{2}}{\sin \frac{A+D+B+E+\theta}{2} \sin \frac{B+E+C+F+\theta}{2} \sin \frac{C+F+A+D+\theta}{2} \sin \frac{\theta}{2}} d\theta$$

where

$$\alpha := \operatorname{atan} \frac{p_\alpha}{q_\alpha} \quad \beta := \operatorname{atan} \frac{p_\beta}{q_\beta}$$

and

$$\begin{aligned}
 p_\alpha &:= \sin(A + B + C + D + E + F) + \sin(A + D) + \sin(B + E) + \sin(C + F) \\
 &\quad + \sin(D + E + F) + \sin(D + B + C) + \sin(A + E + C) + \sin(A + B + F) \\
 q_\alpha &:= -(\cos(A + B + C + D + E + F) + \cos(A + D) + \cos(B + E) + \cos(C + F) \\
 &\quad + \cos(D + E + F) + \cos(D + B + C) + \cos(A + E + C) + \cos(A + B + F)) \\
 p_\beta &:= \sqrt{p_\alpha^2 + q_\alpha^2 - p_\beta^2} = 2\sqrt{-\det G} \\
 q_\beta &:= 2(\sin A \sin D + \sin B \sin E + \sin C \sin F)
 \end{aligned}$$

We note that

$$\begin{aligned}
 (44) \quad p_\alpha^2 + q_\alpha^2 &= p_\beta^2 + q_\beta^2 \\
 &= 8 \left(1 + \ddot{A}\ddot{B}\ddot{C} + \ddot{A}\ddot{E}\ddot{F} + \ddot{D}\ddot{B}\ddot{F} + \ddot{D}\ddot{E}\ddot{C} \right. \\
 &\quad \left. + \ddot{A}\ddot{D}\ddot{B}\ddot{E} + \ddot{B}\ddot{E}\ddot{C}\ddot{F} + \ddot{C}\ddot{F}\ddot{A}\ddot{D} + \overline{ADBE} + \overline{BECF} + \overline{CFAD} \right)
 \end{aligned}$$

Also, with considerable trigonometric effort, one finds that the difference of the numerator and denominator in the logarithm's argument reduces thusly:

$$(45) \quad \cos(\cdot) \cos(\cdot) \cos(\cdot) \cos(\cdot) - \sin(\cdot) \sin(\cdot) \sin(\cdot) \sin(\cdot) = q_\beta - (p_\alpha \sin \theta + q_\alpha \cos \theta)$$

The values $\theta = \alpha \pm \beta$ are roots of this expression and thus are also roots of the integrand.

Unfortunately —unlike in the Euclidean case¹⁸— these formula, in general, resist direct hedronometric re-parameterization in terms of face and pseudo-face areas. (Below, we examine a couple of less-resistant special cases, and we discuss the complexity of the general case.) For the most part, the best we can do currently is invoke the Second Law of Cosines (2) to convert areas into dihedral angle measures, and substitute those measures into the Derevnin-Mednykh formula (43).

8.1. Regular Tetrahedra, $A = B = C = D = E = F$. Since $a = b = c = d = e = f$ and $A = B = C = D = E = F$, we have $dV = -6 a_2 dA$. We can express this differential in terms of angle A , face area X , or pseudo-face area H :

$$(46a) \quad -\frac{dV}{6} = \operatorname{atanh} \sqrt{\frac{3\ddot{A} - 1}{1 - \ddot{A}}} dA = \operatorname{atanh} \sqrt{2\ddot{U}_3 - 1} \frac{dA}{dX} dX = \operatorname{atanh} \sqrt{2\overline{H}_4} \frac{dA}{dH} dH$$

$$(46b) \quad \frac{dA}{dX} = -\frac{\widehat{U}_6}{3\sqrt{1 + 3\ddot{U}_3}} \quad \frac{dA}{dH} = -\frac{\overline{L}_8}{2(3 + 2\ddot{L}_4)}$$

¹⁸The volume, V , of a Euclidean tetrahedron with faces W, X, Y, Z and pseudo-faces H, J, K satisfies a formula bearing a striking resemblance to the Third Law of Cosines in hyperbolic space (3).

$$\begin{aligned}
 81V^4 &= 2W^2X^2Y^2 + 2W^2Y^2Z^2 + 2W^2Z^2X^2 + 2X^2Y^2Z^2 + H^2J^2K^2 \\
 &\quad - H^2(W^2X^2 + Y^2Z^2) - J^2(W^2Y^2 + Z^2X^2) - K^2(W^2Z^2 + X^2Y^2)
 \end{aligned}$$

with $U := \pi - X$ and $L := 2\pi - H$. So

$$(47a) \quad V = -6 \int_{\arccos \frac{1}{3}}^A \operatorname{atanh} \sqrt{\frac{3 \cos \theta - 1}{1 - \cos \theta}} d\theta$$

$$(47b) \quad = 2 \int_0^X \frac{\tan(\pi_6 - \xi_6)}{\sqrt{1 + 3 \cos(\pi_3 - \xi_3)}} \operatorname{atanh} \sqrt{2 \cos(\pi_3 - \xi_3) - 1} d\xi$$

$$(47c) \quad = 3 \int_0^H \frac{\sin(\pi_4 - \eta_8)}{3 + 2 \sin \eta_4} \operatorname{atanh} \sqrt{2 \sin \eta_4} d\eta$$

8.2. Isohedral Right-Corner Tetrahedra, $A = B = C = \pi/2$, $D = E = F$. Limiting ourselves (and our paths of integration) to the universe of right-corner tetrahedra with $A = B = C = \pi/2$, we can assert $dA = dB = dC = 0$; then, with $D = E = F$, we have $dD = dE = dF$, so that $dV = -3 d_2 dD$. We can write:

$$(48a) \quad -\frac{dV}{3} = \operatorname{atanh} \frac{\sqrt{3\ddot{D}^2 - 1}}{\overline{D}} dD = \operatorname{atanh} \sqrt{\overline{X}} \frac{dD}{dX} dX = \operatorname{atanh} \sqrt{2\ddot{U}_3 - 1} \frac{dD}{dW} dW$$

$$(48b) \quad \frac{dD}{dX} = \frac{-\overline{T}_2}{4(1 + \overline{T}_2^2)} \quad \frac{dD}{dW} = \frac{-\widehat{U}_6}{6\sqrt{\ddot{U}_3}}$$

with $T := \pi_2 - X$ and $U := \pi - W$. So

$$(49a) \quad V = -3 \int_{\arccos \sqrt{\frac{1}{3}}}^D \operatorname{atanh} \frac{\sqrt{3 \cos^2 \theta - 1}}{\sin \theta} d\theta$$

$$(49b) \quad = \frac{3}{2} \int_0^X \frac{\sin(\pi_4 - \xi_2)}{1 + \cos^2(\pi_4 - \xi_2)} \operatorname{atanh} \sqrt{\sin \xi} d\xi$$

$$(49c) \quad = \frac{1}{2} \int_0^W \frac{\tan(\pi_6 - \omega_6)}{\sqrt{\cos(\pi_3 - \omega_3)}} \operatorname{atanh} \sqrt{2 \cos(\pi_3 - \omega_3) - 1} d\omega$$

8.3. General Tetrahedra. A key complication in deriving a face-based formula for tetrahedral volume is that our seven hedronometric parameters —faces areas W, X, Y, Z and pseudo-face areas H, J, K — aren't independent (and shouldn't be, as there are only *six* degrees of freedom in determining a tetrahedron). Nevertheless, the "differentialized" Second Law of Cosines (2) converts the Schlafli angular differentials to hedronometric ones:

$$(50a) \quad 4\sqrt{[H_2 Y_2 Z_2]} dA = \overline{H}_2 dH - \frac{\ddot{Z}_2 - \ddot{Y}_2 \ddot{H}_2}{\overline{Y}_2} dY - \frac{\ddot{Y}_2 - \ddot{Z}_2 \ddot{H}_2}{\overline{Z}_2} dZ$$

$$4\sqrt{[H_2 W_2 X_2]} dD = \overline{H}_2 dH - \frac{\ddot{X}_2 - \ddot{W}_2 \ddot{H}_2}{\overline{W}_2} dW - \frac{\ddot{W}_2 - \ddot{X}_2 \ddot{H}_2}{\overline{X}_2} dX$$

while the differentialized Third Law of Cosines (3) expresses the dependency among the hedronometric differentials:

$$(50b) \quad \begin{aligned} \langle W_2 \rangle \overline{W_2} dW + \langle X_2 \rangle \overline{X_2} dX + \langle Y_2 \rangle \overline{Y_2} dY + \langle Z_2 \rangle \overline{Z_2} dZ \\ = \langle H_2 \rangle \overline{H_2} dH + \langle J_2 \rangle \overline{J_2} dJ + \langle K_2 \rangle \overline{K_2} dK \end{aligned}$$

While we hope for a monolithic integral in the spirit of Derevnin-Mednykh that respects the inherent symmetries of the face and pseudo-face area parameters, for now we investigate an asymmetric solution.

8.3.1. *Using an auxiliary parameter.* Consider the universe of tetrahedra with constant W, X, Y, Z, H . Variable pseudo-face areas J and K have a dependency via the Third Law of Cosines (3). We can interpret the Law as a quadratic in \ddot{J}_2 and \ddot{K}_2 that describes an ellipse (rotated by $\pi/4$ from the coordinate axes), allowing us to parameterize the values in terms of a single (and remarkably well-suited) ‘‘polar’’ angle, θ . Translating and rotating the ellipse, we arrive at these relations, where we indicate dependence of J and K on θ with a superscript:

$$(51a) \quad \begin{aligned} \ddot{J}_2^\theta \overline{H_2}^2 = \ddot{W}_2 \ddot{Y}_2 + \ddot{Z}_2 \ddot{X}_2 - \ddot{H}_2 (\ddot{W}_2 \ddot{Z}_2 + \ddot{X}_2 \ddot{Y}_2) \\ + 4 \cos(\theta - H_4) \sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]} \end{aligned}$$

$$(51b) \quad \begin{aligned} \ddot{K}_2^\theta \overline{H_2}^2 = \ddot{W}_2 \ddot{Z}_2 + \ddot{X}_2 \ddot{Y}_2 - \ddot{H}_2 (\ddot{W}_2 \ddot{Y}_2 + \ddot{Z}_2 \ddot{X}_2) \\ - 4 \cos(\theta + H_4) \sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]} \end{aligned}$$

Corresponding formulas for $\langle J_2^\theta \rangle$ and $\langle K_2^\theta \rangle$ reduce nicely

$$(52a) \quad \langle J_2^\theta \rangle \overline{H_2} = 4 \sin(\theta + H_4) \sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]}$$

$$(52b) \quad \langle K_2^\theta \rangle \overline{H_2} = 4 \sin(\theta - H_4) \sqrt{[[H_2 W_2 X_2]] [[H_2 Y_2 Z_2]]}$$

and provide the insight that, given the non-negative nature of each other factor in the products, we must have $\sin(\theta \pm H_4) \geq 0$, so that we can take $H_4 \leq \theta \leq \pi - H_4$.

Abusing notation slightly, we define

$$(53) \quad \langle H_2^\theta \rangle := \ddot{H}_2 + \ddot{J}_2^\theta \ddot{K}_2^\theta - \ddot{W}_2 \ddot{X}_2 - \ddot{Y}_2 \ddot{Z}_2$$

and observe that $\langle H_2^\theta \rangle$ does not appear to admit insightful simplification, so that its behavior with respect to θ remains unclear. Numerical experiments in *Mathematica* suggest that prudent (re-)labeling of face areas can guarantee $\langle H_2^\theta \rangle$ to be strictly positive across the domain determined by $\langle J_2^\theta \rangle$ and $\langle K_2^\theta \rangle$ in the preceding paragraph. The validity of this phenomenon would help streamline Theorem 8.2; unfortunately, proof is elusive, so we state it here as a conjecture:

Conjecture 8.1. *Given a hyperbolic tetrahedron, one can assign W, X, Y, Z to its face areas —and H to the corresponding pseudo-face area— in such a way that $\langle H_2^\theta \rangle > 0$ for all $H_4 < \theta < \pi - H_4$.*

The relations (52) provide compact representations of the trigonometric functions of θ :¹⁹

$$(54a) \quad 4\bar{\theta} \sqrt{[[H_2W_2X_2] [H_2Y_2Z_2]]} = \bar{H}_4(\langle J_2^\theta \rangle + \langle K_2^\theta \rangle)$$

$$(54b) \quad 4\ddot{\theta} \sqrt{[[H_2W_2X_2] [H_2Y_2Z_2]]} = \dot{H}_4(\langle J_2^\theta \rangle - \langle K_2^\theta \rangle)$$

$$(54c) \quad \hat{\theta} = \widehat{H}_4 \frac{\langle J_2^\theta \rangle + \langle K_2^\theta \rangle}{\langle J_2^\theta \rangle - \langle K_2^\theta \rangle}$$

With most face areas constant, the differentials of J^θ and K^θ are straightforward:²⁰

$$(55a) \quad \overline{J_2^\theta H_2^2} dJ^\theta = 8 \sin(\theta - H_4) \sqrt{[[H_2W_2X_2] [H_2Y_2Z_2]]} d\theta$$

$$(55b) \quad \overline{K_2^\theta H_2^2} dK^\theta = -8 \sin(\theta + H_4) \sqrt{[[H_2W_2X_2] [H_2Y_2Z_2]]} d\theta$$

Moreover, the various angular differentials (as in (50a)) simplify. Of course, dA and dD vanish outright; otherwise, we have, for instance,

$$(55c) \quad dB = \frac{\overline{J_2^\theta} dJ^\theta}{4\sqrt{[[J_2^\theta Z_2 X_2]]}} = \frac{2\sqrt{[[H_2W_2X_2] [H_2Y_2Z_2]]} \sin(\theta - H_4)}{\overline{H_2^2} \sqrt{[[J_2^\theta Z_2 X_2]]}} d\theta$$

We may therefore give this formula for the Schafli volume differential (42):

$$(56a) \quad dV = \frac{\sqrt{[[H_2W_2X_2] [H_2Y_2Z_2]]}}{\overline{H_2^2}} \left(\begin{array}{l} -\sin(\theta - H_4) \left(\frac{b^\theta}{\sqrt{[[J_2^\theta Z_2 X_2]]}} + \frac{e^\theta}{\sqrt{[[J_2^\theta W_2 Y_2]]}} \right) \\ +\sin(\theta + H_4) \left(\frac{c^\theta}{\sqrt{[[K_2^\theta X_2 Y_2]]}} + \frac{f^\theta}{\sqrt{[[K_2^\theta W_2 Z_2]]}} \right) \end{array} \right) d\theta$$

for non-negative $b^\theta, c^\theta, e^\theta, f^\theta$ satisfying

$$(56b) \quad \widehat{b_2^\theta}^2 = 4 \frac{[[J_2^\theta Z_2 X_2]]}{\langle H_2^\theta \rangle} \frac{\sin(\theta + H_4)}{\sin(\theta - H_4)} \quad \widehat{e_2^\theta}^2 = 4 \frac{[[J_2^\theta W_2 Y_2]]}{\langle H_2^\theta \rangle} \frac{\sin(\theta + H_4)}{\sin(\theta - H_4)}$$

$$\widehat{c_2^\theta}^2 = 4 \frac{[[K_2^\theta X_2 Y_2]]}{\langle H_2^\theta \rangle} \frac{\sin(\theta - H_4)}{\sin(\theta + H_4)} \quad \widehat{f_2^\theta}^2 = 4 \frac{[[K_2^\theta W_2 Z_2]]}{\langle H_2^\theta \rangle} \frac{\sin(\theta - H_4)}{\sin(\theta + H_4)}$$

Now, note that our constant areas make the variable aspect of \widehat{G} , the tetrahedron's Gram determinant (see (12a)), proportional to $\langle H_2^\theta \rangle \langle J_2^\theta \rangle \langle K_2^\theta \rangle$, which is in turn proportional to $\langle H_2^\theta \rangle \sin(\theta + H_4) \sin(\theta - H_4)$. Consequently, \widehat{G} vanishes—indicating a degenerate tetrahedron—at the endpoints H_4 and $\pi - H_4$ of the bounds on θ imposed after (52); in the absence of proof of Conjecture 8.1, we recognize the possibility of degenerate tetrahedra determined by some $\theta = \theta^*$ between these endpoints.

Interpreting the above appropriately, we find that we can compute the volume of a tetrahedron via a hedronometric formula that integrates through our universe of tetrahedra

¹⁹Squaring and adding the sine and cosine relations re-confirms equation (6).

²⁰With (52), these are evidently consistent with the differentialized Third Law of Cosines (50b).

with constant W, X, Y, Z, H , from the degenerate form (corresponding to $\theta = H_4$, or possibly $\theta = \theta^*$) to our target (corresponding to the particular θ satisfying (54), with $J^\theta = J$ and $K^\theta = K$).

Theorem 8.2. *A hyperbolic tetrahedron with face areas W, X, Y, Z and corresponding pseudo-face areas H, J, K has volume, V , given by*
(57)

$$V = \frac{\sqrt{[H_2 W_2 X_2][H_2 Y_2 Z_2]}}{\overline{H_2}^2} \int_{\Phi}^{\Theta} \left(\begin{array}{l} -\sin(\theta - H_4) \left(\frac{b^\theta}{\sqrt{[J_2^\theta Z_2 X_2]}} + \frac{e^\theta}{\sqrt{[J_2^\theta W_2 Y_2]}} \right) \\ +\sin(\theta + H_4) \left(\frac{c^\theta}{\sqrt{[K_2^\theta X_2 Y_2]}} + \frac{f^\theta}{\sqrt{[K_2^\theta W_2 Z_2]}} \right) \end{array} \right) d\theta$$

with

- non-negative $b^\theta, c^\theta, e^\theta, f^\theta$ satisfying (56b);
- $H_4 \leq \Theta \leq \pi - H_4$, such that $\theta = \Theta$ satisfies (54) with $J^\theta = J$ and $K^\theta = K$;
- Φ is the largest root of $\langle H_2^\theta \rangle \langle J_2^\theta \rangle \langle K_2^\theta \rangle = 0$ such that $H_4 \leq \Phi \leq \Theta$.²¹

9. WHAT IS THE PYTHAGOREAN THEOREM FOR RIGHT-CORNER 4-SIMPLICES?

In Euclidean any-dimensional space, our understanding of the Pythagorean Theorem for right-corner simplices is complete: always, *the square of the content of the hypotenuse is equal to the sum of the squares of the contents of the legs*. In hyperbolic geometry, our knowledge seems embarrassingly limited:

Dimension 2 : $\cosh \text{hyp} = \cosh \text{leg}_1 \cosh \text{leg}_2$

Dimension 3 : $\cos \frac{\text{hyp}}{2} = \cos \frac{\text{leg}_1}{2} \cos \frac{\text{leg}_2}{2} \cos \frac{\text{leg}_3}{2} - \sin \frac{\text{leg}_1}{2} \sin \frac{\text{leg}_2}{2} \sin \frac{\text{leg}_3}{2}$

Dimension 4+ : $??? = ???$

Consider the case of a right-corner simplex in hyperbolic 4-space: as Section 8 indicates, any relation between the volume of the simplex’s hypotenuse-tetrahedron and the volumes of its leg-tetrahedra involves comparing complicated integrals. Even the most-special of special cases (below) defies this author’s attempts at comprehension.

²¹By Conjecture 8.1, appropriate face and pseudo-face labeling would allow us to take $\Phi = H_4$.

9.1. Isosceles Right-Corner 4-Simplices. Consider right-corner 4-simplex such that each of its four congruent legs are isohedral right-corner tetrahedra determined by dihedral angles $(A, B, C, D, E, F) = (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, P, P, P)$; its hypotenuse is a regular tetrahedron with congruent dihedral angles (Q, Q, Q, Q, Q, Q) . One can verify that

$$(58) \quad \frac{1}{3} \leq \cos^2 P = \chi = \cos Q \leq \frac{1}{2}$$

for a parameter χ that we use to write tantalizingly-similar formulas for the volume H of the hypotenuse (via (47a)) and volume L of each leg (via (49a)):

$$(59a) \quad H := -3 \int_{\arccos \sqrt{\frac{1}{3}}}^{\arccos \sqrt{\chi}} \operatorname{atanh} \sqrt{\frac{3 \cos^2 \theta - 1}{1 - \cos^2 \theta}} d\theta = \frac{3}{2} \int_{\frac{1}{3}}^{\chi} \frac{1}{\sqrt{t(1-t)}} \operatorname{atanh} \sqrt{\frac{3t-1}{1-t}} dt$$

$$(59b) \quad L := -6 \int_{\arccos \frac{1}{3}}^{\arccos \chi} \operatorname{atanh} \sqrt{\frac{3 \cos \theta - 1}{1 - \cos \theta}} d\theta = 6 \int_{\frac{1}{3}}^{\chi} \frac{1}{\sqrt{1-t^2}} \operatorname{atanh} \sqrt{\frac{3t-1}{1-t}} dt$$

Although Pythagoras beckons, a connection between these volumes remains elusive. Indeed, the rather exotic—but not unfamiliar²²—values for $\chi = 1/2$ (defining a simplex with a quadrupally-asymptotic hypotenuse and triply-asymptotic legs)

$$(60) \quad H^* := \sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{\pi k}{3} = 1.01494 \dots \quad L^* := \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{\pi k}{2} = 0.45978 \dots$$

suggest that the corresponding Pythagorean Theorem is quite non-trivial.

²² L^* is half of Catalan's constant, and H^* is known, e.g., as the maximum of the Clausen function, Cl_2 .

A. STANDARD NOTATION

This note (in the tradition of most notes on hedronometry by this author) reserves much—in fact, just-over half— of the Latin alphabet for specific notational use, as described below. Generally, lower-case letters indicate “trigonometric” elements (sides and planar angles, and their measures), while upper-case letters indicate “hedronometric” elements (faces and dihedral angles, and their measures); the lower- and upper-case counterparts of a letter indicate elements in a natural correspondence.

We avoid the use of integer subscripts as indices; rather, we reserve them for compact representation of fractions in formulas. (See Section A.2.)

Note. Without fear of confusion, a symbol represents both a geometric element and its measure. (E.g., edge a has length a ; angle B has measure B ; face Z has area Z .)

A.1. Tetrahedral Elements ($a, b, c, d, e, f; W, X, Y, Z; H, J, K$). A tetrahedron’s edges a, b, c concur at a vertex and oppose respective edges d, e, f . The dihedral angles along the edges are, respectively, A, B, C and D, E, F .

The tetrahedron’s faces are

$$W := \triangle def \quad X := \triangle dbc \quad Y := \triangle aec \quad Z := \triangle abf$$

When discussing “right-corner” tetrahedra, we take edges a, b, c to be mutually-orthogonal, and angles A, B, C to be right angles. This makes W the figure’s “hypotenuse-face” and X, Y, Z its right-triangular “leg-faces”.

We sometimes (as in Section 4) assign symbols w, x, y, z to the altitudes to faces W, X, Y, Z , but other times revert using these symbols for generic quantities and coordinates in the traditional way.

With regard to pseudo-elements²³ (and their measures), we make the following associations between various pairs of traditional elements and trios comprising a pseudo-face (H, J, K), a pseudo-altitude (h, j, k), and a twist angle (τ_h, τ_j, τ_k):

$$\begin{aligned} a, d &\leftrightarrow A, D \leftrightarrow (Y, Z), (W, X) \leftrightarrow H, h, \tau_h \\ b, e &\leftrightarrow B, E \leftrightarrow (Z, X), (W, Y) \leftrightarrow J, j, \tau_j \\ c, f &\leftrightarrow C, F \leftrightarrow (X, Y), (W, Z) \leftrightarrow K, k, \tau_k \end{aligned}$$

A.2. Trigonometry. To remove visual clutter from trigonometric expressions, we adopt this “Morse Code” (and hat) notation:

$$\begin{aligned} \ddot{x} &:= \cosh x & \bar{x} &:= \sinh x & \hat{x} &:= \tanh x & , \text{ for lengths } x \\ \ddot{\theta} &:= \cos \theta & \bar{\theta} &:= \sin \theta & \hat{\theta} &:= \tan \theta & , \text{ for angles } \theta \\ \ddot{X} &:= \cos X & \bar{X} &:= \sin X & \hat{X} &:= \tan X & , \text{ for areas } X \end{aligned}$$

Context should make clear whether hyperbolic or circular functions are intended.

²³See footnotes 1 and 11 for definitions of *pseudo-face* and *pseudo-altitude*.

“Half-measures” (half-angles, half-lengths, half-areas) and even “quarter-measures” appear so often in formulas that writing (and reading) them as fractions can be a strain. For typographic simplicity, we write “ x_n ” for “ x/n ”.

Finally, we define the “Heronic sine and cosine products”²⁴ via

$$(61a) \quad \begin{aligned} \llbracket x, y, z \rrbracket &:= \overline{(x_2 + y_2 + z_2)} \overline{(-x_2 + y_2 + z_2)} \overline{(x_2 - y_2 + z_2)} \overline{(x_2 + y_2 - z_2)} \\ &= \frac{1}{4} (2\ddot{x}\ddot{y}\ddot{z} - \ddot{x}^2 - \ddot{y}^2 - \ddot{z}^2 + 1) \end{aligned}$$

$$(61b) \quad \begin{aligned} \llbracket x, y, z \rrbracket &:= \overline{\overline{\overline{\overline{(x_2 + y_2 + z_2)} (-x_2 + y_2 + z_2) (x_2 - y_2 + z_2) (x_2 + y_2 - z_2)}}}} \\ &= \frac{1}{4} (2\ddot{x}\ddot{y}\ddot{z} + \ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2 - 1) \end{aligned}$$

These take their name and symbology from the so-called and -symbolized (by me) algebraic Heronic product

$$(62) \quad [x, y, z] := (x_2 + y_2 + z_2) (-x_2 + y_2 + z_2) (x_2 - y_2 + z_2) (x_2 + y_2 - z_2)$$

that gives Heron’s formula for the square of the area of a *Euclidean* triangle with edges x, y, z . This note occasionally uses (62) to abbreviate the characteristic four-part product.²⁵

We’ll note the “side-angle-side” form of the Heronic (hyperbolic) sine product of edges x, y, z of a hyperbolic triangle, with angles θ, ϕ, ψ opposite respective sides x, y, z :

$$(63) \quad \llbracket x, y, z \rrbracket = \frac{1}{4} \overline{\overline{\overline{\overline{y^2 z^2 \theta^2}}}} = \frac{1}{4} \overline{\overline{\overline{\overline{z^2 x^2 \phi^2}}}} = \frac{1}{4} \overline{\overline{\overline{\overline{x^2 y^2 \psi^2}}}}$$

This equation inspires the Law of “Side-Angle-Side-Altitude” Sines (Theorem 4.1).

A.3. Matrices, Determinants, Conjugation Ratio, and Minors (G, M). Define the traditional (“angle”) Gram matrix, G , and conjugate (“edge”) Gram matrix, g , with rows and columns indexed by faces W, X, Y, Z (in order).²⁶

$$(64) \quad G := \begin{bmatrix} 1 & -\ddot{D} & -\ddot{E} & -\ddot{F} \\ -\ddot{D} & 1 & -\ddot{C} & -\ddot{B} \\ -\ddot{E} & -\ddot{C} & 1 & -\ddot{A} \\ -\ddot{F} & -\ddot{B} & -\ddot{A} & 1 \end{bmatrix} \quad g := \begin{bmatrix} -1 & -\ddot{a} & -\ddot{b} & -\ddot{c} \\ -\ddot{a} & -1 & -\ddot{f} & -\ddot{e} \\ -\ddot{b} & -\ddot{f} & -1 & -\ddot{d} \\ -\ddot{c} & -\ddot{e} & -\ddot{d} & -1 \end{bmatrix}$$

²⁴The equivalence of factored and expanded forms holds for both circular and hyperbolic functions.

²⁵See, for instance, (8a) and (27c). Note that, because I incorporate the halving of arguments in building the factors (to provide uniformity with the sine and cosine products), I must *double* arguments — or else multiply the result by 16 — when I simply want to use the “[.]” notation to compress these four-fold products.

²⁶The angles in a row (column) of G *surround* the index-face, while remaining angles *oppose* that face; conjugately, the edges in a row (column) of g *oppose* the index-face, while the remaining edges *surround* it.

The determinants of these matrices are as follows:

$$(65a) \quad \widehat{G} := \det G = \begin{pmatrix} 1 - \ddot{A}^2 - \ddot{B}^2 - \ddot{C}^2 - \ddot{D}^2 - \ddot{E}^2 - \ddot{F}^2 \\ -2\ddot{A}\ddot{B}\ddot{C} - 2\ddot{A}\ddot{E}\ddot{F} - 2\ddot{D}\ddot{B}\ddot{F} - 2\ddot{D}\ddot{E}\ddot{C} \\ +\ddot{A}^2\ddot{D}^2 + \ddot{B}^2\ddot{E}^2 + \ddot{C}^2\ddot{F}^2 \\ -2\ddot{A}\ddot{D}\ddot{B}\ddot{E} - 2\ddot{B}\ddot{E}\ddot{C}\ddot{F} - 2\ddot{C}\ddot{F}\ddot{A}\ddot{D} \end{pmatrix} \leq 0$$

$$(65b) \quad \widehat{g} := \det g = \begin{pmatrix} 1 - \ddot{a}^2 - \ddot{b}^2 - \ddot{c}^2 - \ddot{d}^2 - \ddot{e}^2 - \ddot{f}^2 \\ +2\ddot{d}\ddot{b}\ddot{c} + 2\ddot{a}\ddot{e}\ddot{c} + 2\ddot{a}\ddot{b}\ddot{f} + 2\ddot{d}\ddot{e}\ddot{f} \\ +\ddot{a}^2\ddot{d}^2 + \ddot{b}^2\ddot{e}^2 + \ddot{c}^2\ddot{f}^2 - 2\ddot{a}\ddot{d}\ddot{b}\ddot{e} - 2\ddot{a}\ddot{d}\ddot{c}\ddot{f} - 2\ddot{b}\ddot{e}\ddot{c}\ddot{f} \end{pmatrix} \leq 0$$

and we define (what I call) the *conjugation ratio*²⁷ thusly:

$$(66) \quad \Gamma := \frac{\widehat{G}}{\widehat{g}}$$

For face-indices P and Q , define the (PQ) -th cofactors, M_{PQ} and m_{PQ} :

$$(67) \quad M_{PQ} := (-1)^{P+Q} \text{minor}_{PQ} G \quad m_{PQ} := (-1)^{P+Q} \text{minor}_{PQ} g$$

Thus,

$$(68a) \quad M_{WX} = \overline{A}^2 \ddot{D} + \ddot{A} (\ddot{B}\ddot{E} + \ddot{C}\ddot{F}) + \ddot{B}\ddot{F} + \ddot{C}\ddot{E}$$

$$(68b) \quad m_{WX} = -\ddot{a}\ddot{d}^2 + \ddot{d} (\ddot{b}\ddot{e} + \ddot{c}\ddot{f}) - (\ddot{b}\ddot{f} + \ddot{c}\ddot{e})$$

When $P = Q$, we use a single index. For instance,

$$(69a) \quad M_W = -2\ddot{A}\ddot{B}\ddot{C} - \ddot{A}^2 - \ddot{B}^2 - \ddot{C}^2 + 1 = -4[A, B, C]$$

$$(69b) \quad m_W = -2\ddot{d}\ddot{e}\ddot{f} + \ddot{d}^2 + \ddot{e}^2 + \ddot{f}^2 - 1 = -4[[d, e, f]] = -16\overline{W}_2^2 \ddot{d}_2^2 \ddot{e}_2^2 \ddot{f}_2^2$$

We note that Mednykh and Pashkevich [3, Proposition 4] provide cofactor formulas for the lengths of a tetrahedron's edges and altitudes:

$$(70a) \quad \ddot{a}^2 = \frac{M_{WX}^2}{M_W M_X} \quad \ddot{d}^2 = \frac{M_{YZ}^2}{M_Y M_Z} \quad \overline{w}^2 = \frac{-\widehat{G}}{M_W} = \frac{\widehat{g}}{m_W}$$

Likewise, we have cofactor formulas for dihedral angles:

$$(70b) \quad \ddot{D}^2 = \frac{m_{WX}^2}{m_W m_X} \quad \ddot{A}^2 = \frac{m_{YZ}^2}{m_Y m_Z}$$

²⁷Note the use of upper-case gamma as a reminder that the numerator of the conjugation ratio contains the upper-cased —that is, *angle-based*— determinant, \widehat{G} . I'm tempted to introduce “ γ ” to denote the reciprocal ratio, just for the sake of completeness.

B. COORDINATES

In coordinatized hyperbolic space,²⁸ let $P(x_p, y_p, z_p)$ be such that

- z_p measures the *signed* length of a perpendicular dropped from P to a point P_{xy} in the xy -plane;
- y_p measures the *signed* length of a perpendicular dropped from P_{xy} to a point P_x on the x -axis; and
- x_p measures the *signed* distance from P_x to the origin.

Every coordinate system has a distance formula, and here's the one for this context:

Theorem B.1 (The Distance Formula in Coordinatized Hyperbolic Space). *The distance, $|AB|$, from point $A(x_a, y_a, z_a)$ to point $B(x_b, y_b, z_b)$ is given by*

$$(71) \quad \cosh |AB| = \ddot{z}_a \ddot{z}_b (\ddot{y}_a \ddot{y}_b (\ddot{x}_a \ddot{x}_b - \overline{x_a x_b}) - \overline{y_a y_b}) - \overline{z_a z_b}$$

We wish to describe a coordinatization of a tetrahedron. Let the tetrahedron have elements with standard²⁹ edge, (pseudo)face, and pseudo-altitude labels, and let vertices P, Q, R, S be opposite respective faces W, X, Y, Z . We align pseudo-altitude h with the x -axis and an adjacent edge PQ with the y -axis; say that the pseudo-altitude separates PQ into segments of length a_P and a_Q , and RS into segments of length d_R and d_S . We write

$$P(0, -a_P, 0) \quad Q(0, a_Q, 0) \quad R(h, -y_R, -z_R) \quad S(h, y_S, z_S)$$

where

$$\begin{aligned} \tanh y_R &= \tanh d_R \cos \tau_h & \tanh y_S &= \tanh d_S \cos \tau_h \\ \sinh z_R &= \sinh d_R \sin \tau_h & \sinh z_S &= \sinh d_S \sin \tau_h \end{aligned}$$

for τ_h the angular *twist* of edge RS out of the xy -plane.

This arrangement, and the Distance Formula, provide these formulas for edge lengths.³⁰

$$(72) \quad \begin{aligned} \ddot{a} &= \cosh |PQ| = \ddot{a}_P \ddot{a}_Q + \overline{a_P a_Q} & \ddot{d} &= \cosh |RS| = \ddot{d}_R \ddot{d}_S + \overline{d_R d_S} \\ \ddot{b} &= \cosh |PR| = \ddot{a}_P \ddot{d}_R \ddot{h} - \overline{a_P d_R} \ddot{\tau}_h & \ddot{e} &= \cosh |SQ| = \ddot{a}_Q \ddot{d}_S \ddot{h} - \overline{a_Q d_S} \ddot{\tau}_h \\ \ddot{c} &= \cosh |PS| = \ddot{a}_P \ddot{d}_S \ddot{h} + \overline{a_P d_S} \ddot{\tau}_h & \ddot{f} &= \cosh |QR| = \ddot{a}_Q \ddot{d}_R \ddot{h} + \overline{a_Q d_R} \ddot{\tau}_h \end{aligned}$$

With these, the conjugate Gram determinant, \hat{g} , (see (65b)) collapses from over a dozen terms to just one:

$$(73) \quad \hat{g} = -\overline{a^2 d^2 h^2 \tau_h^2}$$

²⁸Here, we enlist x, y, z as symbols for standard coordinates instead of tetrahedral altitudes.

²⁹See Appendix A.

³⁰For instance,

$$\ddot{b} = \ddot{z}_R (\ddot{a}_P \ddot{y}_R (\ddot{h} - 0) - \overline{a_P y_R}) - 0 = \ddot{a}_P (\ddot{y}_R \ddot{z}_R) \ddot{h} - \overline{a_P (y_R z_R)} = \ddot{a}_P \ddot{d}_R \ddot{h} - \overline{a_P d_R} \ddot{\tau}_h$$

where the simplifications $\ddot{y}_R \ddot{z}_R = \ddot{d}_R$ and $\overline{y_R z_R} = \overline{d_R \tau_h}$ follow from the trigonometry of the right triangle with hypotenuse d_R , legs y_R and z_R , and angle τ_h .

whence, all un-squared sines being non-negative while \widehat{g} being non-positive,

$$(74a) \quad \bar{a} \bar{d} \bar{h} \bar{\tau}_h = \sqrt{-\widehat{g}}$$

Also, one readily verifies this identity:

$$(74b) \quad \bar{a} \bar{d} \ddot{h} |\ddot{\tau}_h| = \left| \ddot{c} \ddot{f} - \ddot{b} \ddot{e} \right|$$

where the supplementary ambiguity of τ_h necessitates the use of absolute values.

With a bit of work, one can extract the sub-segment lengths a_P , etc, writing

$$(75a) \quad \sinh 2a_P = \frac{2\bar{a}(m_{WX} + \ddot{a}m_X)}{\sqrt{-[2\bar{a}\bar{d}, 2(\ddot{b}\ddot{e} - \ddot{c}\ddot{f}), 2i\sqrt{-\widehat{g}}]}}$$

$$(75b) \quad \sinh 2a_Q = \frac{2\bar{a}(m_{WX} + \ddot{a}m_W)}{\sqrt{-[2\bar{a}\bar{d}, 2(\ddot{b}\ddot{e} - \ddot{c}\ddot{f}), 2i\sqrt{-\widehat{g}}]}}$$

so that

$$(76) \quad \frac{\sinh 2a_P}{\sinh 2a_Q} = \frac{m_{WX} + \ddot{a}m_X}{m_{WX} + \ddot{a}m_W}$$

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