HERON-LIKE HEDRONOMETRIC RESULTS FOR TETRAHEDRAL VOLUME

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Heron of Alexandria (c. 10 CE – 70 CE) devised this formula for a triangle's area A in terms of its edge-lengths a, b, c:

$$16A^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c) \tag{0.1}$$

Such a formula raises an obvious question: Is there a "dimensionally-enhanced" analogue for a *tetrahedron's volume* in terms of its *face-areas*? A little thought (and Figure 1) reveals the equally-obvious answer: No. Simply, a tetrahedron's shape admits *six* degrees of freedom —e.g., one can determine a tetrahedron by its six edges, or by three edges and three angles at a vertex— so a mere *four* face-areas cannot account for this amount of variation. Face-areas do not determine a tetrahedron's volume.

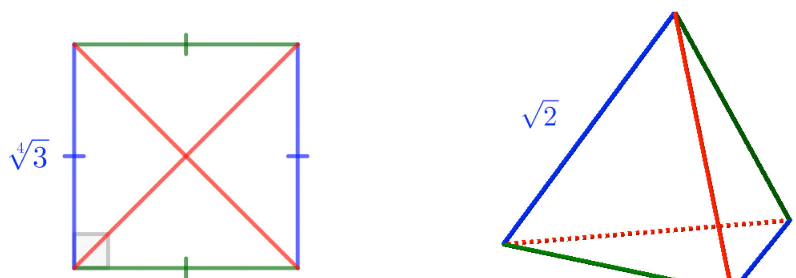

FIGURE 1. A "flat" tetrahedron coinciding with a planar square of edge-length $\sqrt[4]{3}$, and a regular tetrahedron of edge-length $\sqrt{2}$, have matching face-areas (namely, all $\sqrt{3}/2$), but distinct volumes (namely, zero and non-).

Consequently, any *hedronometric* (face-based) volume formula must incorporate one of two compromise options: (1) involve more faces(?!), or (2) accommodate fewer tetrahedra. This note addresses both prospects, presenting (in Section 2) the Pseudo-Heron formula, which adds the areas of the tetrahedron's three *pseudofaces* into the computational mix; and (in Section 3) the Heron Quartic, a polynomial relating volume and just four (non-pseudo)face-areas, but only for so-called "perfect" tetrahedra (see the Appendix).

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1. Preliminaries

1.1. Notation. Define the following elements of tetrahedron $OABC$ (see Figure 2):

$$(1.1)\qquad \begin{array}{ccc} a := |\overline{OA}| & b := |\overline{OB}| & c := |\overline{OC}| \\ d := |\overline{BC}| & e := |\overline{CA}| & f := |\overline{AB}| \\ \\ \alpha := \angle BOC & \beta := \angle COA & \gamma := \angle AOB \\ \\ X := |\triangle BOC| & Y := |\triangle COA| & Z := |\triangle AOB| \\ & W := |\triangle ABC| & \end{array}$$

Also, let $\angle A$, $\angle B$, etc. —or simply A, B, etc.— indicate the dihedral angles between faces meeting along edges a, b, etc.

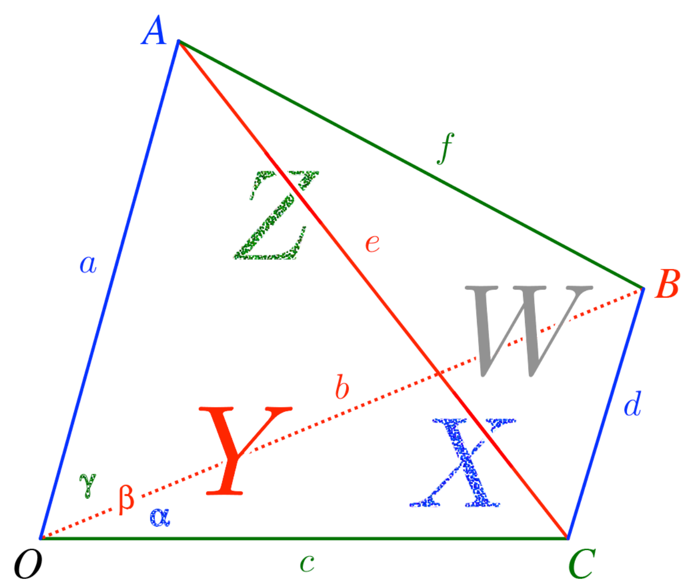

Figure 2. A tetrahedron with edges a, b, c, d, e, f, faces W, X, Y, Z, and face-angles α, β, γ. Our pseudofaces (not shown) are associated with pairs of opposite edges: H with $\{a, d\}$, J with $\{b, e\}$, K with $\{c, f\}$.

1.2. Hedronometric Fundamentals. Hedronometry, as the dimensionally-enhanced trigonometry of tetrahedra, features a few not-well-known relations that we'll review here.

Theorem 1 (The Law of Cosines for Adjacent Dihedral Angles; "Law of Adjacent Cosines")**.**

$$W^2 = X^2 + Y^2 + Z^2 - 2YZ\cos A - 2ZX\cos B - 2XY\cos C \qquad (1.2)$$

Note that, for a "right-corner" tetrahedron —i.e., one with three mutually-perpendicular edges meeting at a vertex— the Law of Adjacent Cosines reduces to a hedronometric Pythagorean relation, which has come to be known by a different name.[1]

Corollary 1 (de Gua's Theorem)**.** *For a right-corner tetrahedron with hypotenuse-face W,*

$$W^2 = X^2 + Y^2 + Z^2 \tag{1.3}$$

Importantly, de Gua's theorem is generally only one-way: a tetrahedron satisfying (1.3) need not have a right corner, unless it is also *perfect.*

We define a *pseudoface* of a tetrahedron as the quadrilateral shadow of the figure in a direction perpendicular to a pair of opposite edges, the projections of those edges being the diagonals of the shadow. Our pseudofaces H, J, K are determined by respective edge-pairs $\{a,d\}$, $\{b,e\}$, $\{c,f\}$, with areas calculated via a Bretschneider-like formula; for instance,

$$16H^2 = 4a^2d^2 - \left(\left(b^2+e^2\right) - \left(c^2+f^2\right)\right)^2 \tag{1.4}$$

Pseudofaces are what make hedronometry *work*, via these remaining relations:

Theorem 2 (The Law of Cosines for Opposite Dihedral Angles; "Law of Opposite Cosines")**.**

$$\begin{aligned} Y^2+Z^2-2YZ\cos A &= H^2 = W^2+X^2-2WX\cos D \\ Z^2+X^2-2ZX\cos B &= J^2 = W^2+Y^2-2WY\cos E \\ X^2+Y^2-2XY\cos C &= K^2 = W^2+Z^2-2WZ\cos F \end{aligned} \tag{1.5}$$

Theorem 3 (The Sum-of-Squares Identity)**.**

$$W^2+X^2+Y^2+Z^2 = H^2+J^2+K^2 \tag{1.6}$$

Among other things, the sum-of-squares serves as the key dependency that restricts a tetrahedron's *seven* (pseudo)face-areas to only *six* degrees of freedom, the precise number needed to determine the tetrahedron's shape.

We'll list a couple of consequences of the Laws of Cosines and sum-of-squares. The first we'll need later; the second is more of a curiosity (although, see Equation (4.3)).

Corollary 2 (Miscellaneous Formulas)**.**

$$\begin{aligned} J^2K^2-(WX-YZ)^2 &= 2\,WXYZ\,(1+\cos A\cos D-\cos B\cos E-\cos C\cos F) \\ K^2H^2-(WY-ZX)^2 &= 2\,WXYZ\,(1-\cos A\cos D+\cos B\cos E-\cos C\cos F) \\ H^2J^2-(WZ-XY)^2 &= 2\,WXYZ\,(1-\cos A\cos D-\cos B\cos E+\cos C\cos F) \end{aligned} \tag{1.7a}$$

[1]After Jean Paul de Gua de Malves, who published the result in 1783. Howard Eves [2] notes: "The theorem, however, had been known to Descartes (1596–1650) and his contemporary J[ohann] Faulhaber (1580–1635). It is a special case of a more general theorem that [Charles de] Tinseau [d'Amondans] had presented to the Paris Academy of Sciences in 1774."

$$(H+J+K)(-H+J+K)(H-J+K)(H+J-K)$$
$$= (-W+X+Y+Z)(W-X+Y+Z)(W+X-Y+Z)(W+X+Y-Z)$$
$$-\ 8WXYZ(1+\cos A\cos D+\cos B\cos E+\cos C\cos F) \tag{1.7b}$$

Finally, we have a counterpart to trigonometry's vertex-centric area formula:

Theorem 4 (The Vertex-Centric Volume Formula)**.** *A tetrahedron's volume, V, satisfies*

$$81V^4 = 4X^2Y^2Z^2\ \left(\ 1-2\cos A\cos B\cos C-\cos^2 A-\cos^2 B-\cos^2 C\ \right) \tag{1.8a}$$

This is merely a hedronometric re-packaging of the better-known formula that features edges and face-angles[2]

$$36V^2 = a^2b^2c^2\ \left(\ 1+2\cos\alpha\cos\beta\cos\gamma-\cos^2\alpha-\cos^2\beta-\cos^2\gamma\ \right) \tag{1.8b}$$

It serves as the launch point for our quest to compute volume from areas alone.

2. The Pseudo-Heron Volume Formula

Using the Law of Opposite Cosines (1.5) to eliminate cosines from the vertex-centric volume formula (1.8a), we easily derive a Heron-like relation involving volume, face-areas, and pseudoface-areas. The sum-of-squares helps us wrangle the symbols into a tidy form.

Theorem 5 (The Pseudo-Heron Volume Formula)**.** *A tetrahedron's volume, V, satisfies*

$$\begin{aligned} 81V^4 \ =\ & H^2J^2K^2 - 2\,(WX-YZ)\,(WY-ZX)\,(WZ-XY) \\ & - H^2\,(WX-YZ)^2 - J^2\,(WY-ZX)^2 - K^2\,(WZ-XY)^2 \end{aligned} \tag{2.1}$$

The formula has an immediate corollary for tetrahedra whose faces all agree in area:

Corollary 3. *The volume of an equihedral tetrahedron is given by*

$$9V^2 = HJK \tag{2.2}$$

Apart from that, there isn't a great deal to say about the Pseudo-Heron volume formula, except to note some interesting alternative forms. For instance, here it is as a determinant:

$$81V^4 \ =\ \begin{vmatrix} H^2 & XY-WZ & ZX-WY \\ XY-WZ & J^2 & YZ-WX \\ ZX-WY & YZ-WX & K^2 \end{vmatrix} \tag{2.3}$$

One may notice that the cofactors of the diagonal elements are precisely the left-hand expressions of (1.7a). We can highlight those expressions in the volume formula itself.

$$\begin{aligned} 81V^4 \ =\ & -2H^2J^2K^2 - 2\,(WX-YZ)\,(WY-ZX)\,(WZ-XY) \\ & + H^2\left(J^2K^2-(WX-YZ)^2\right) + J^2\left(K^2H^2-(WY-ZX)^2\right) \\ & + K^2\left(H^2J^2-(WZ-XY)^2\right) \end{aligned} \tag{2.4}$$

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[2] We can convert from one to another using the Laws of Cosines from spherical trigonometry:

$$\cos\alpha = \cos\beta\cos\gamma+\sin\beta\sin\gamma\cos A \qquad \cos A = -\cos B\cos C+\sin B\sin C\cos\alpha$$

Replacing those expressions with their cosined equivalents via (1.7a) ultimately gives this:

$$\begin{aligned}(2.5)\quad 4\cdot 81V^4 = &\left(W^2+X^2-Y^2-Z^2\right)\left(W^2-X^2-Y^2+Z^2\right)\left(W^2-X^2+Y^2-Z^2\right)\\ &+\left(-H^2+J^2+K^2\right)\left(H^2-J^2+K^2\right)\left(H^2+J^2-K^2\right)\\ &+16WXYZ\left(H^2\cos A\cos D+J^2\cos B\cos E+K^2\cos C\cos F\right)\end{aligned}$$

The next version of the volume formula appears in other notes by this author.

$$\begin{aligned}(2.6)\quad 81V^4 \;=\; &2W^2X^2Y^2+2W^2X^2Z^2+2W^2Y^2Z^2+2X^2Y^2Z^2+H^2J^2K^2\\ &-H^2\left(W^2X^2+Y^2Z^2\right)-J^2\left(W^2Y^2+Z^2X^2\right)-K^2\left(W^2Z^2+X^2Y^2\right)\end{aligned}$$

It, too, has a determinant form, although it's not as direct a match as above.

$$(2.7)\quad -32\cdot 81V^4 \;=\; \begin{vmatrix} 4W^2 & H^2 & J^2 & K^2 & 1\\ H^2 & 4X^2 & K^2 & J^2 & 1\\ J^2 & K^2 & 4Y^2 & H^2 & 1\\ K^2 & J^2 & H^2 & 4Z^2 & 1\\ 1 & 1 & 1 & 1 & 0\end{vmatrix} \mod \begin{pmatrix} W^2+X^2+Y^2+Z^2\\ -H^2-J^2-K^2\end{pmatrix}$$

3. The Heron Quartic for Perfect Tetrahedra

By restricting our attention to a convenient sub-family of tetrahedra, we can reduce the degrees of freedom they exhibit to a manageable four, making a Heron-like volume formula in (non-pseudo)face-areas feasible. But which sub-family deserves our focus?

Discussions of Heron's formula often reference the fact that the result is equivalent to de Gua's theorem (1.3) applied to a (possibly-imaginary) right-corner tetrahedron whose hypotenuse-face is the target triangle. The details of that argument are left to the reader; our take-away is the useful characterization of the figures of interest.

Our favored tetrahedral sub-family consists of those figures that serve as hypotenuse-cells of right-corner simplices in *four-dimensional space*. A more-accessible characterization may be that *each pair of opposite edges determines a pair of orthogonal vectors*. The family of perfect tetrahedra includes two significant sub-families —the regular (Platonic) tetrahedra and the right-corner tetrahedra— as well as the volume-maximizing instances of tetrahedra with given sets of faces-areas (see Theorem 4.3). Some details of perfect tetrahedra appear in this note's Appendix.

Now, to derive a relation between a perfect tetrahedron's volume and its face-areas, "all we have to do" is eliminate H, J, K from the Pseudo-Heron volume formula (2.1). We manage this with the help of the sum-of-squares identity (1.6), the Appendix's hedronometric characterization of perfection (4.2c), and a computer algebra system —such as *Mathematica*'s `Resultant[]` function— to perform the tedious symbol-manipulation.

Unfortunately, volume and face-areas are entangled in a relation with an exceedingly-unwieldy explicit form. We'll settle for an only-modestly-unwieldy implicit presentation.

Theorem 6 (The Heron Quartic for Perfect Tetrahedral Volume)**.** *Face-areas W, X, Y, Z, and volume V, of a perfect tetrahedron satisfy*

$$
\begin{aligned}
0 \;=\; 27\,U^4 + U^3 \;&\left(\, \sigma_\star\sigma_1 + 8\,(4\sigma_1\sigma_2 - 27\sigma_3)\,\right) \\
- U^2 \;&\left(\, \sigma_\star^2\sigma_2 + 12\sigma_\star\,(3\sigma_1\sigma_3 - 28\sigma_4) - 48\,\left(9\sigma_3^2 - 16\sigma_2\sigma_4\right)\,\right) \\
+ U \;&\left(\, \sigma_\star^3\sigma_3 + 40\sigma_\star^2\sigma_1\sigma_4 - 576\sigma_\star\sigma_3\sigma_4 + 1536\sigma_1\sigma_4^2\,\right) \\
- \quad &\sigma_4\left(\, \sigma_\star^2 - 64\sigma_4\,\right)^2
\end{aligned}
\tag{3.1}
$$

where $U := 81V^4$, and the σ_i are symmetric polynomials *in the squares of the face-areas:*[3]

$$
\begin{aligned}
\sigma_1 &:= W^2 + X^2 + Y^2 + Z^2 \\
\sigma_2 &:= W^2X^2 + W^2Y^2 + W^2Z^2 + X^2Y^2 + X^2Z^2 + Y^2Z^2 \\
\sigma_3 &:= W^2X^2Y^2 + W^2X^2Z^2 + W^2Y^2Z^2 + X^2Y^2Z^2 \\
\sigma_4 &:= W^2X^2Y^2Z^2 \\
\sigma_\star &:= 4\sigma_2 - \sigma_1^2 = -W^4 - X^4 - Y^4 - Z^4 + 2W^2X^2 + 2W^2Y^2 + \cdots + 2Y^2Z^2
\end{aligned}
$$

3.1. **Too Many Roots.** Of course, a quartic polynomial has *four* roots, so Theorem 6 asserts only that a (non-degenerate) perfect tetrahedron's volume corresponds to *one of the (positive) roots* of its Heron Quartic. There's always a viable root; this is confirmed by the fact that the polynomial's coefficient sequence exhibits an odd number of sign changes,[4] so that the Descartes Rule of Signs guarantees an odd number of positive roots.

Somewhat embarrassingly, this author is (so far) unable to declare that there's always an *unambiguous* choice of root among as many as three candidates. The Rule of Signs doesn't help, because the sign behavior of the complicated middle coefficients is somewhat inscrutable. Even so, computer sampling across various face-areas has consistently generated Heron Quartics with three non-positive roots and a single viable positive root.[5]

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[3]The special grouping $\sigma_\star$ hearkens-back to the three-variable elements appearing in Heron's formula:

$$16\,|\triangle abc|^2 = \; -a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2$$

During attempts to simplify the coefficients, $\sigma_\star$ seemed to "want" to appear —one may notice the progression $\sigma_\star\sigma_1$, $-\sigma_\star^2\sigma_2$, $\sigma_\star^3\sigma_3$, $-\sigma_\star^4\sigma_4$ among the coefficients— but whether this speaks to some structure within the Quartic is unclear. In fact, $\sigma_\star$ itself may not be the most-appropriate grouping: as indicated in (3.3), expressions *adjacent* to $\sigma_\star$ have oh-so-tantalizing factorizations.

$$
\begin{aligned}
\sigma_\star + 8WXYZ &= \quad(-W + X + Y + Z)(W - X + Y + Z)(W + X - Y + Z)(W + X + Y - Z) \\
\sigma_\star - 8WXYZ &= -(\;\; W + X + Y + Z)(W + X - Y - Z)(W - X + Y - Z)(W - X - Y + Z)
\end{aligned}
$$

At least $\sigma_\star$ helps reduce *some* of the coefficient clutter, and we'll take whatever help we can get: expanded, the polynomial has over five hundred terms.

[4]The fourth-power coefficient is positive, while the constant term is negative (or zero, in which case, the *linear* term is negative (or zero, in which case, ...)).

[5]Interestingly, the results were consistent even across face-areas that violated the tetrahedral "Triangle Inequality" $W \leq X + Y + Z$, etc.

The Quartic's discriminant is worth noting:

$$\begin{gathered}\left(W^2-X^2\right)^2\left(W^2-Y^2\right)^2\left(W^2-Z^2\right)^2\left(X^2-Y^2\right)^2\left(Y^2-Z^2\right)^2\left(Z^2-X^2\right)^2 \\ \left(\,\sigma_\star^4-16\sigma_\star^3\sigma_2+288\sigma_\star^2\sigma_1\sigma_3-1152\sigma_\star\sigma_4(5\sigma_1^2+4\sigma_2)+55296\sigma_4(\sigma_1\sigma_3-2\sigma_4)\,\right)^3\end{gathered} \tag{3.2}$$

From this, we learn that a tetrahedron's Heron Quartic has a multiple root if any two face-areas match (that is, when the figure is "bisohedral"); we show later that those roots are non-positive. Otherwise, numerical sampling again provides a consistently-positive discriminant; when, as discussed above, there is at least one positive root and/or one negative root, we know that all roots are real.

3.2. **Obvious degeneracies, and not-so-obvious non-degeneracies.** Expressing the Heron Quartic in terms of symmetric polynomials obscures some obvious zero-volume cases. Once de-sigma-tized, the Quartic's constant term factors as

$$\begin{gathered}W^2X^2Y^2Z^2\left(W+X+Y+Z\right)^2 \\ \left(-W+X+Y+Z\right)^2\left(W-X+Y+Z\right)^2\left(W+X-Y+Z\right)^2\left(W+X+Y-Z\right)^2 \\ \left(W+X-Y-Z\right)^2\left(W-X+Y-Z\right)^2\left(W-X-Y+Z\right)^2\end{gathered} \tag{3.3}$$

If any of the factors vanish, then $U=0$ is root of the polynomial; in almost-all cases, the conditions under which a factor vanishes imply a degenerate tetrahedron, so that $U=0$ is *the* root that corresponds to the volume of that tetrahedron. For example,

- $W=0$ and $W+X+Y+Z=0$ imply one or four degenerate faces. Volume is zero.
- $W=X+Y+Z$ implies the degenerate case of the tetrahedral analogue of the Triangle Inequality (eg, $W\leq X+Y+Z$), so that the figure is "flat", with faces X, Y, Z subdividing face W (via edges meeting at W's orthocenter). Volume is zero.
- $W+X=Y+Z$ *does not* imply a degenerate figure. (Regular tetrahedra fall into this case, as do doubly-bisohedrals.) On the contrary, the condition implies a *non-degenerate* figure, since a perfect "flat" tetrahedron has edges in an orthocentric triangular configuration, with face-areas related as in the previous case. Volume is non-zero.

3.3. **Special Cases.** Here's a brief survey of circumstances under which the Heron Quartic simplifies. Throughout, we see that at most one root is ever viable for computing volume.

3.3.1. *Equihedral Tetrahedron* ($W=X=Y=Z$). A perfect equihedral tetrahedron is necessarily regular, with four equilateral faces. Its Heron Quartic reduces thusly:

$$0=U^3\left(\,27U-2^6W^6\,\right)\qquad\to\qquad 3^7V^4=2^6W^6 \tag{3.4}$$

3.3.2. *Trisohedral Tetrahedron* ($W;X=Y=Z$). A perfect trisohedral tetrahedron is a pyramid with an equilateral base and congruent isosceles lateral faces. Its Heron Quartic:

$$0=\left(27U-W^2\left(W^2-9X^2\right)^2\right)\left(U+X^2\left(W^2-X^2\right)^2\right)^3\quad\to\quad 3^7V^4=W^2\left(W^2-9X^2\right)^2 \tag{3.5}$$

3.3.3. *Bisohedral Tetrahedron* ($W = X; Y, Z$). A perfect bisohedral tetrahedron's Heron Quartic factors thusly:

(3.6a)

$$0 = \left(U + X^2(Y^2 - Z^2)^2 \right)^2 \cdot \left(\begin{array}{c} 27U^2 + U(4X^2 - Y^2 - Z^2)(4X^2 - Y^2 - 6YZ - Z^2)(4X^2 - Y^2 + 6YZ - Z^2) \\ -Y^2Z^2(2X + Y + Z)^2(2X - Y + Z)^2(2X + Y - Z)^2(2X - Y - Z)^2 \end{array} \right)$$

The squared factor is non-zero for a non-degenerate tetrahedron, while the latter quadratic has a single non-negative root, giving

$$\begin{aligned} 2 \cdot 3^7 V^4 = &- (4X^2 - Y^2 - Z^2)(4X^2 - Y^2 - 6YZ - Z^2)(4X^2 - Y^2 + 6YZ - Z^2) \\ &+ \left((4X^2 - Y^2 - Z^2)^2 + 12Y^2Z^2\right)^{3/2} \end{aligned} \tag{3.6b}$$

In the case of a *doubly*-bisohedral tetrahedron, with $Y = Z$, the expression reduces slightly:

$$3^7 V^4 = 16(2X^2 - Y^2)(X^2 - 2Y^2)(X^2 + Y^2) + 2\left(X^4 - X^2Y^2 + Y^4\right)^{3/2} \tag{3.6c}$$

3.3.4. *Pythagorean/deGuan Tetrahedron* ($W^2 = X^2 + Y^2 + Z^2$). In general, a Pythagorean tetrahedron need not have a right corner; however, a *perfect* Pythagorean tetrahedron *must*. The Heron Quartic factors as

$$0 = \left(U - 4X^2Y^2Z^2 \right) (\dots) \tag{3.7a}$$

which admits only one positive root,[6] yielding the relation consistent with (1.8a).

$$9V^2 = 2XYZ \tag{3.7b}$$

3.4. **Perfect Volume is Maximal.** A perfect tetrahedron's orthogonal opposite edges seem as though they're trying to embrace as much space as possible; as it happens, they succeed (see [3]):

Theorem 7 (Gerber, 1975)**.** *A perfect tetrahedron maximizes volume for a given set of face-areas.*

Gerber's proof of this result —which actually covers *arbitrary-dimensional* orthocentric simplices— is fairly involved, but fairly elementary, using vector techniques and basic calculus. For completeness, we'll use basic calculus to prove maximality from our hedronometric relations. Given our current inability to guarantee that the Heron Quartic has a sole viable volume-related root, we'll state a weaker result:

[6] For the curious, the "$(\dots)$" factor is

$$\begin{aligned} &U^3 + U^2 \left(16(X^6 + Y^6 + Z^6) + 30(X^4Y^2 + X^2Y^4 + X^2Y^4 + Y^4Z^2 + X^2Z^4 + Y^2Z^4) + 15X^2Y^2Z^2 \right) \\ &\quad + U \left(\begin{array}{c} 8(\, X^8(Y^2 - Z^2)^2 + Y^8(Z^2 - X^2)^2 + Z^8(X^2 - Y^2)^2 \,) + 15(\, X^6Y^6 + Y^6Z^6 + Z^6X^6 \,) \\ +X^2Y^2Z^2(\, X^4Y^2 + X^4Z^2 + X^2Y^4 + Y^4Z^2 + X^2Z^4 + Y^2Z^4 \,) + 30X^4Y^4Z^4 \end{array} \right) \\ &\quad + \left(X^2 + Y^2 + Z^2\right)(-XY + YZ + ZX)^2 (XY - YZ + ZX)^2 (XY + YZ - ZX)^2 \end{aligned}$$

The coefficients are evidently non-negative, so that this polynomial admits no non-negative roots.

Theorem 8. *A perfect tetrahedron* locally *maximizes volume for a given set of face-areas.*

Proof. Using the sum-of-squares identity to eliminate K^2 from the Pseudo-Heron volume formula (most-conveniently, in the form of (2.6)), we consider the function $L(h,j)$ in variables $h := H^2$ and $j := J^2$:

$$\begin{aligned} L(h,j) &= hj(W^2+X^2+Y^2+Z^2-h-j) \\ &\quad - h(WX-YZ)^2 - j(WY-ZX)^2 + (h+j)(WZ-XY)^2 \end{aligned} \tag{3.8a}$$

For fixed W, X, Y, Z, this L covers the non-constant terms of Pseudo-Heron formula; maximizing L maximizes volume. Now, the stationary points of L satisfy $\partial L/\partial h = \partial L/\partial j = 0$:

$$\begin{aligned} 0 = L_h &= j\left(W^2+X^2+Y^2+Z^2-h-j\right) - hj - (WX-YZ)^2 + (WZ-XY)^2 \\ = L_j &= h\left(W^2+X^2+Y^2+Z^2-h-j\right) - hj - (WY-ZX)^2 + (WZ-XY)^2 \end{aligned} \tag{3.8b}$$

One sees, upon replacing H^2, J^2, and K^2, that this condition is equivalent to the hedronometric characterization of perfection (4.2c), which says exactly that *perfect tetrahedral volume is "critical"*. To prove that it's (locally) *maximal*, we need more derivatives.

$$\begin{aligned} L_{hh} &= -2j &&= -2J^2 \\ L_{jj} &= -2h &&= -2H^2 \\ L_{hj} = L_{jh} &= -2h-2j+W^2+X^2+Y^2+Z^2 &&= -H^2-J^2+K^2 \end{aligned} \tag{3.8c}$$

Since L_{hh} is negative, and since the Hessian determinant,

$$L_{hh}L_{jj} - L_{hj}^2 = (H+J+K)(-H+J+K)(H-J+K)(H+J-K) \tag{3.8d}$$

is positive (by (4.3)), (local) maximality follows from the Second Derivative Test. □

It's rather satisfying that the linchpin of this argument is the "Heronic product" (as in 0.1) of pseudoface-areas. This seems to hint at an as-yet-hidden structure in our Heron-like volume relations, while also testifying to the fundamental influence pseudofaces have over their tetrahedron's nature.

4. Appendix: Hedronometric Aspects of Perfection.

A *perfect* tetrahedron is one in which each edge is orthogonal to its opposite:

$$\overline{OA} \perp \overline{BC} \qquad \overline{OB} \perp \overline{CA} \qquad \overline{OC} \perp \overline{AB} \tag{4.1}$$

A tetrahedron satisfying any two orthogonality conditions satisfies all three. Perfection, then, reduces the degrees of freedom in the variation of such a tetrahedron from *six* to *four*, which is why face-areas alone suffice to determine a perfect tetrahedron's shape.

The lore of perfect tetrahedra —also known as "orthocentric" tetrahedra, for their concurrent altitudes— stretches back to the late 1700s, occasionally repeating itself. In 1934, N. A. Court observed [1]: *It is impossible to examine the literature on the orthocentric tetrahedron without being struck by the fact that the same properties recur time and again, being rediscovered by various authors quite independently.* This author hopes that the current discussion of *hedronometric* properties expands scholarship in this (*ahem*) area.

There are a few metric characterizations of perfection. For example,

$$a^2 + d^2 \;=\; b^2 + e^2 \;=\; c^2 + f^2 \quad \big(= \; 4m^2\,\big) \tag{4.2a}$$

which, here, follows from the Bretschneider-like formula for pseudoface area (1.4) when those areas reduce to $H = ad/2$, $J = be/2$, $K = cf/2$; the common sum involves m, the length of a *bimedian* segment joining the midpoints of two opposite edges. Further, writing $4WXYZ = (-H^2 + W^2 + X^2)(-H^2 + Y^2 + Z^2)$, and replacing d, e, f with $\sqrt{4m^2 - a^2}$, $\sqrt{4m^2 - b^2}$, $\sqrt{4m^2 - c^2}$, gives a product symmetric in a, b, c. This implies a (known?) dihedral characterization of perfection

$$\cos A \cos D \;=\; \cos B \cos E \;=\; \cos C \cos F \tag{4.2b}$$

that, with (1.7a), provides a (new?) purely *hedronometric* characterization:

$$J^2K^2 - (WX - YZ)^2 = K^2H^2 - (WY - ZX)^2 = H^2J^2 - (WZ - XY)^2 \tag{4.2c}$$
$$\big(\; = 2WXYZ(1 - \cos A \cos D) \;\; \big)$$

In Section 3, we use that last relation to derive the Heron Quartic for perfect tetrahedral volume (3.1). To prove that perfect volume is (locally) maximal across tetrahedra with given face areas (Theorem 8), we use the following result:

Lemma 9. *A perfect tetrahedron's pseudoface-areas H, J, K, volume V, and circumradius r satisfy*

$$36\,V^2r^2 \;=\; (H + J + K)(-H + J + K)(H - J + K)(H + J - K) \tag{4.3}$$

This follows immediately from substituting $ad = 2H$, $be = 2J$, $cf = 2K$, into a formula valid for all tetrahedra

$$2^6 3^2\,V^2r^2 = (ad + be + cd)(-ad + be + cf)(ad - be + cf)(ad + be - cf)$$

References

[1] Court, N. A., Notes on the Orthocentric Tetrahedron. *The American Mathematical Monthly.* Vol. 41, No. 8 (Oct., 1934), pp. 499–502. `http://jstor.org/stable/2300415`

[2] Eves, Howard W. *Great Moments in Mathematics (before 1650).* Mathematical Association of America, 1983. p. 37.

[3] Gerber, L., The Orthocentric Simplex as an Extreme Simplex. *Pacific Journal of Mathematics.* Vol. 56, No. 1 (Nov., 1975), pp. 97–111. `https://msp.org/pjm/1975/56-1/pjm-v56-n1-p09-s.pdf`