HERON-LIKE STRATEGIES FOR NON-EUCLIDEAN TETRAHEDRAL VOLUME

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The note [3] presents the Pseudo-Heron Formula for the volume of a (Euclidean) tetrahedron in terms of the areas of its four faces and three pseudo-faces. It also introduces the Heron Quartic that encodes the volume of a “perfect” tetrahedron within a fourth-degree polynomial whose coefficients are expressed in terms of the areas of the four faces. This note offers a preliminary reconnaissance exercise in the quest for analogues of these results for tetrahedra in hyperbolic space.

Briefly, a hyperbolic tetrahedron’s volume is a function of the measures of its six dihedral angles. Basic “hedronometry” provides the necessary (and familiar) bridges between these angle measures and face-and-pseudo-face areas, guaranteeing a Pseudo-Heron result of some form. Likewise, “perfection” introduces dependencies between (proper) faces and pseudo-faces, allowing us to eliminate the latter from consideration and compute the volume of perfect tetrahedra in proper Heronic fashion: using their face areas alone.

At least, that’s the idea.

Unfortunately, the nature of the key hyperbolic volume function (it’s an integral) makes the consequent Pseudo-Heron result less than satisfying, as we cannot give a straightforward explicit (or implicit) formula; and, while perfection does admit elimination of pseudo-face elements, the relations involve 24th-degree polynomials. At best, our results are reduced to strategies on the order of “substitute appropriate solutions of some equations into the established hyperbolic volume formula”.

The final section of this note investigates the possibility of deriving a custom integral for at least the Pseudo-Heron case, though the author has not yet engaged in any significant investigation of that approach.

1. Notation

Denote the faces (and, without confusion, their areas) of a tetrahedron by $W$, $X$, $Y$, and $Z$; let its pseudo-faces (and areas) be $H$, $J$, and $K$. Finally, name the dihedral angles (and their measures)

$$A := \angle YZ \quad B := \angle ZX \quad C := \angle XY$$
$$D := \angle WX \quad E := \angle WY \quad F := \angle WZ$$

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To conserve space in long formulas, we employ the following “Morse Code” for cosines and sines of areas or angles:

\[
\hat{X} := \cos X \quad \bar{X} := \sin X
\]

Moreover, for typographical convenience, as many formulas involving face areas feature trigonometric functions of half-areas, we write, eg, “\(\frac{X}{2}\)” for “\(\frac{\hat{X}}{2}\)”. Thus, \(\hat{X}_2 := \cos \frac{X}{2}\).

2. Two Hedronometric Formulas

We pull two results from [4] for reference purposes.

The first result forms the bridge between dihedral angle information and face-and-pseudo-face area information (and it serves to formally define the pseudo elements). We introduce the convention that \(H, J, \text{ and } K\) be bounded by \(2\pi\), although this consideration does not enter into discussion until Section 5, where it is invoked to settle a sign ambiguity.\(^1\)

**Theorem 2.1** (The Second-and-a-Halfth (“2.5th”) Law of Cosines).

\[
\begin{align*}
\hat{Y}_2 \hat{Z}_2 + \bar{Y}_2 \bar{Z}_2 \bar{A} &= \hat{H}_2 & = & \hat{W}_2 \hat{X}_2 + \bar{W}_2 \bar{X}_2 \bar{D} \\
\hat{Z}_2 \hat{X}_2 + \bar{Z}_2 \bar{X}_2 \bar{B} &= \hat{J}_2 & = & \hat{W}_2 \hat{Y}_2 + \bar{W}_2 \bar{Y}_2 \bar{E} \\
\hat{X}_2 \hat{Y}_2 + \bar{X}_2 \bar{Y}_2 \bar{C} &= \hat{K}_2 & = & \hat{W}_2 \hat{Z}_2 + \bar{W}_2 \bar{Z}_2 \bar{F}
\end{align*}
\]

The second result is described as a “symmetric, face-agnostic” Law of Cosines, which here is dubbed the Third Law\(^2\).

\(^1\)As described in [5], the pseudo-faces of a Euclidean tetrahedron are the projections of the figure into planes parallel to pairs of opposite edges. No geometric interpretation of non-Euclidean pseudo-faces is (yet) known; they merely serve to make the 2.5th Law of Cosines resemble the laws of cosines relating edges and angles of hyperbolic triangles. (As a point of history, Euclidean pseudo-faces first appeared as formal definitions in a Law of Cosines.) Observe that as, for instance, dihedral angle \(A\) ranges from 0 to \(\pi\), the value of the expression for \(\cos \frac{H}{2}\) ranges from \(\cos \frac{Y-Z}{2}\) to \(\cos \frac{Y+Z}{2}\). It is not unreasonable, then, to declare that \(H\) falls between \(Y-Z\) and \(Y+Z\); under this agreement, the fact that hyperbolic triangles have a maximum area of \(\pi\) provides an absolute maximum value of \(2\pi\) on \(H\).

\(^2\)This is, in fact, an analogue of the Sum of Squares formula for Euclidean tetrahedra (from [3]): with each cosine term expanded as a power series, and then with all but the lowest-power—in this case, fourth-power—terms discarded, the Third Law reduces to

\[
0 \approx \frac{1}{64} (W^2 + X^2 + Y^2 + Z^2 - H^2 - J^2 - K^2)^2
\]

Thus, as a tetrahedron approaches infinitesimal size, the relation among its face areas approaches the Euclidean result, \(W^2 + X^2 + Y^2 + Z^2 = H^2 + J^2 + K^2\).
**Theorem 2.2** (The Third Law of Cosines).

\[
0 = 1 - \tilde{W}_2^2 - \tilde{X}_2^2 - \tilde{Y}_2^2 - \tilde{Z}_2^2 - 4\tilde{W}_2\tilde{X}_2\tilde{Y}_2\tilde{Z}_2
- \tilde{H}_2^2 - \tilde{J}_2^2 - \tilde{K}_2^2 - 2\tilde{H}_2\tilde{J}_2\tilde{K}_2
+ 2\tilde{H}_2 \left( \tilde{W}_2\tilde{X}_2 + \tilde{Y}_2\tilde{Z}_2 \right) + 2\tilde{J}_2 \left( \tilde{W}_2\tilde{Y}_2 + \tilde{Z}_2\tilde{X}_2 \right) + 2\tilde{K}_2 \left( \tilde{W}_2\tilde{Z}_2 + \tilde{X}_2\tilde{Y}_2 \right)
\]

3. **The Volume Formula, and the Pseudo-Heron Strategy**

Derevnin and Mednykh [1] offer this formula for the volume of a (compact) tetrahedron with dihedral angles \(A, B, C, D, E, F\):

\[
(1) \quad V = -\frac{1}{4} \int_{\theta_0}^{\theta_0+\phi} \log \frac{\cos \frac{A+B+C+\theta}{2} \cos \frac{A+E+F+\theta}{2} \cos \frac{D+B+F+\theta}{2} \cos \frac{D+E+C+\theta}{2}}{\sin \frac{A+D+B+E+\theta}{2} \sin \frac{A+D+C+F+\theta}{2} \sin \frac{B+E+C+F+\theta}{2} \sin \frac{C+D+E+F+\theta}{2}} d\theta
\]

where

\[
\theta_0 := \tan^{-1} \frac{k_2}{k_1} \quad \phi := \tan^{-1} \frac{k_4}{k_3}
\]

and

\[
k_1 := -\left( \cos(A + B + C + D + E + F) + \cos(A + D) + \cos(B + E) + \cos(C + F) + \cos(D + E + F) + \cos(D + B + C) + \cos(A + E + C) + \cos(A + B + F) \right)
\]

\[
k_2 := \sin(A + B + C + D + E + F) + \sin(A + D) + \sin(B + E) + \sin(C + F) + \sin(D + E + F) + \sin(D + B + C) + \sin(A + E + C) + \sin(A + B + F)
\]

\[
k_3 := 2 \left( \sin A \sin D + \sin B \sin E + \sin C \sin F \right)
\]

\[
k_4 := \sqrt{k_1^2 + k_2^2 - k_3^2}
\]

Importantly, the \(k_4\) value is closely tied to the tetrahedron’s Gram matrix, \(G\):

\[
k_4^2 = -4 \det \begin{pmatrix}
1 & -\cos A & -\cos B & -\cos F \\
-\cos A & 1 & -\cos C & -\cos E \\
-\cos B & -\cos C & 1 & -\cos D \\
-\cos F & -\cos E & -\cos D & 1
\end{pmatrix} = -4 \det G
\]

Note that \(\det G\) (and \(k_4\)) is non-zero for non-degenerate tetrahedra.\(^3\)

\(^3\)If \(k_4 = 0\), then \(\phi = 0\) or \(\phi\) involves an indeterminate fraction requiring \(k_3 = 0\). In the first case, the limits of integration in the volume formula match, yielding \(V = 0\). In the second case, we must have at least three angles of measure 0.
Now, the Derevnin-Mednykh integral, coupled with the 2.5th Law of Cosines, amounts to a Pseudo-Heron Formula, as we have all angle references converted into expressions involving areas of faces and pseudo-faces.

Unfortunately, just “plugging in” doesn’t yield attractive results. When expanded into trig functions of individual dihedral angles, the expressions for $k_1$, $k_2$, and $k_3$ involve not just cosines, but (un-squared) sines; at best, the 2.5th Law would have us express these using square roots, as in

$$
\sin A = \sqrt{1 + 2\bar{H}_2\bar{Y}_2\bar{Z}_2 - \bar{H}_2^2 - \bar{Y}_2^2 - \bar{Z}_2^2}
$$

There seems to be little benefit, then, in converting the $k$s into face-and-pseudo-face form. Better to treat this as a strategy, instead of a formula:

**Theorem 3.1 (The Pseudo-Heron Strategy).** To compute the volume of a hyperbolic tetrahedron from the areas of its faces and pseudo-faces, use the 2.5th Law of Cosines to determine the measures of the dihedral angles $A$, $B$, $C$, etc. (via their cosines), then substitute these measures into the Derevnin-Mednykh formula.

We’ll close this section with the observations that the “all cosines” nature of $k_4$ leads to a not-unpleasant face-and-pseudo-face form:

$$
k_4 = \frac{4}{W_2X_2Y_2Z_2} \cdot \sqrt{\frac{\left(\bar{H}_2 + \bar{J}_2\bar{K}_2 - \bar{W}_2\bar{X}_2 - \bar{Y}_2\bar{Z}_2\right) \left(\bar{J}_2 + \bar{K}_2\bar{H}_2 - \bar{W}_2\bar{Y}_2 - \bar{Z}_2\bar{X}_2\right) \left(\bar{K}_2 + \bar{H}_2\bar{J}_2 - \bar{W}_2\bar{Z}_2 - \bar{X}_2\bar{Y}_2\right)}{\left(\bar{H}_2\bar{J}_2\bar{K}_2 - \bar{W}_2\bar{X}_2\bar{Y}_2 - \bar{Z}_2\bar{X}_2\bar{Y}_2\right)}}
$$

Perhaps the lesson of $k_4$ is that the whole of the “Pseudo-Heron Strategy” somehow admits reduction to a more compact form. Or perhaps not.

### 4. When Perfection Isn’t Perfect

A *perfect* tetrahedron\(^5\) is one in which each pair of opposite edges is orthogonal. The family of perfect tetrahedra includes right-corner tetrahedra (with three orthogonal edges meeting at a vertex) and regular tetrahedra (with equilateral faces). Perfection induces geometric dependencies that reduce the degrees of freedom in such a tetrahedron from six to four, so that face areas alone may characterize the figure. In Euclidean space, this leads to the Heron Quartic (see \cite{3}), a polynomial that allows us to compute the figure’s volume from its face areas; in hyperbolic space, perfection gets us as far as a 12th-degree polynomial with who-knows-how-many extraneous roots. Perfection *does* have an simplifying effect on the Gram matrix, however. We will review some salient properties of perfect tetrahedra here.

\(^4\)This form does not arise simply from a straight substitution of the dihedral angle cosines with their face-and-pseudo-face equivalents. Doing *that* yields an unfactorable 96-term expression. The compact form comes from reducing the expansion *modulo* the Third Law of Cosines.

\(^5\)Also known as an *orthogonal* tetrahedron.
Let $\alpha$, $\beta$, and $\gamma$ be the face-angles —in $X$, $Y$, and $Z$, respectively— having a common vertex, $O$. Let $a := OP$, $b := OQ$, and $c := OR$ be edges, with $a$ opposite $\alpha$, etc. We may assume that $\alpha$ is not a right angle. Therefore, the perpendicular from $R$ meets $OQ$ at a point $T \neq O$. Perfection requires that $RTP \perp OQ$, which implies that $T$ is also the foot of the perpendicular from $P$ to $OQ$. By the trigonometry of hyperbolic triangles,

$$\tanh |OT| = \tanh c \cos \alpha = \tanh a \cos \gamma$$

so that $\cos \alpha / \tanh a = \cos \gamma / \tanh c$. Symmetry provides that we have a value $m$ such that

$$\cos \alpha / \tanh a = \cos \beta / \tanh b = \cos \gamma / \tanh c = 1 / m$$

Writing $\bar{x}$ for $\cosh x$ and $\bar{x}$ for $\sinh x$, we incorporate (2) into the hyperbolic Law of Cosines formula for $d := QR$.

\[ \bar{d} = \bar{b}c - \bar{b} \bar{c} \bar{\alpha} = \bar{b}c - mb\bar{\beta} \cdot \bar{m} \bar{\gamma} \cdot \bar{\alpha} = \bar{b}c \cdot \left( 1 - m^2 \bar{\alpha} \bar{\beta} \bar{\gamma} \right) \]

This leads to another symmetric result.

\[ \bar{a} \bar{d} - \bar{b} \bar{e} = \bar{c} \bar{f} = \bar{a} \bar{b}c \cdot \left( 1 - m^2 \bar{\alpha} \bar{\beta} \bar{\gamma} \right) \]

Indeed, hyperbolic and Euclidean hedronometry share a common characterization of perfection, namely:

6The Euclidean analogue of this relation is

\[ \frac{\cos \alpha}{a} = \frac{\cos \beta}{b} = \frac{\cos \gamma}{c} = \frac{1}{m} \]

7The Euclidean counterpart is

\[ a^2 + d^2 = b^2 + e^2 = c^2 + f^2 = m^2 \left( \bar{a}^2 + \bar{b}^2 + \bar{c}^2 - 2\bar{\alpha} \bar{\beta} \bar{\gamma} \right) \]

8The interconnectedness of results (3) and (4) is evident in, for instance, this relation involving the Gram matrix, $G$:

\[ \bar{a} \bar{d} - \bar{b} \bar{e} = \left( \bar{A} \bar{D} - \bar{B} \bar{E} \right) \det G \]

For maximum symmetry, we can write

\[ \frac{\bar{a} \bar{d} - \bar{b} \bar{e}}{AD - BE} = \frac{\bar{b} \bar{e} - \bar{c} \bar{f}}{BE - CF} = \frac{\bar{c} \bar{f} - \bar{a} \bar{d}}{CF - AD} = \det G \]

with the understanding that, if a non-degenerate tetrahedron’s measurements zero-out the numerator of a component fraction, then (as $G \neq 0$) they must also zero the corresponding denominator. The analogue of this relation in Euclidean space is

\[ \frac{(a^2 + d^2) - (b^2 + e^2)}{AD - BE} = \frac{(b^2 + e^2) - (c^2 + f^2)}{BE - CF} = \frac{(c^2 + f^2) - (a^2 + d^2)}{CF - AD} = -9V^2 / 8WXYZ \]
\[ \cos A \cos D = \cos B \cos E = \cos C \cos F \]

From the 2.5th Law of Cosines, this ultimately implies that our hyperbolic faces and pseudo-faces are related thusly (introducing a “symmetric” parameter, \( M \)):

\[ \begin{align*}
\hat{H}^2 - \hat{H} \left( \hat{W}\hat{X} + \hat{Y}\hat{Z} \right) &= \hat{J}^2 - \hat{J} \left( \hat{W}\hat{Y} + \hat{Z}\hat{X} \right) = \hat{K}^2 - \hat{K} \left( \hat{W}\hat{Z} + \hat{X}\hat{Y} \right) =: M
\end{align*} \]

In [3], the counterpart of this relation, along with the Sum of Squares identity, and the Pseudo-Heron formula for volume, gave rise to the Heron Quartic: we used the four equations to eliminate occurrences of three pseudo-face area quantities, leaving a polynomial involving only face areas and volume. As mentioned, the path to a similar result in hyperbolic space is unavailable.

Short of a volume formula, we can provide relations that express the dihedral angle measures \( A, B, C, \) etc, in terms of \( \hat{W}, \hat{X}, \hat{Y}, \) and \( \hat{Z}, \) but each relation is a product of two 12th-degree polynomials that defy simplification efforts. Given the length of these polynomials (on the order of thousands of terms), we won’t display them here, but will provide a recipe for generating them with a computer algebra system such as Mathematica.

For maximum symmetry, we set a preliminary goal to be a polynomial expressing the value \( M \) from (5) in terms of proper-face areas. From there, converting to polynomials for the pseudo-face areas, and then for dihedral angles, will be straightforward.

Now, we gather together our relevant relations: equation (5), contributing three polynomials, and the Third Law of Cosines:

\[
\begin{align*}
polyH &= H^2 - H \left( W X + Y Z \right) - M \\
polyJ &= J^2 - J \left( W Y + Z X \right) - M \\
polyK &= K^2 - K \left( W Z + X Y \right) - M \\
l0c3 &= 1 - W^2 - X^2 - Y^2 - Z^2 - 4 W X Y Z \\
&\quad - H^2 - J^2 - K^2 - 2 H J K + 2 H \left( W X + Y Z \right) + 2 J \left( W Y + Z X \right) + 2 K \left( W Z + X Y \right)
\end{align*}
\]

We can make our symbol-manipulator’s task a bit less labor-intensive with a few definitions to reduce symbol clutter.

\[
\begin{align*}
S := 1 - W^2 - X^2 - Y^2 - Z^2 - 4 W X Y Z
\end{align*}
\]

The transformed polynomials now feature the variables we wish to eliminate.

\[
\begin{align*}
polyH &= H^2 - H S_X - M \\
polyJ &= J^2 - J S_Y - M \\
polyK &= K^2 - K S_Z - M \\
l0c3 &= S - H^2 - J^2 - K^2 - 2 H J K + 2 H S_X + 2 J S_Y + 2 K S_Z
\end{align*}
\]

Note that we can simplify the Third Law of Cosines even further:

\[
l0c3 = S + H^2 - J^2 - K^2 - 2 H J K - 6 M
\]
At this point, we let our computer take over, repeatedly invoking, for instance, Mathematica’s\footnote{Herons} \textit{Resultant} function to eliminate variables $H_2$, $J_2$, and $K_2$ from the four polynomials. The result (ant?) is a 12th degree polynomial in $M$—with over a thousand terms involving $S$, $S_x$, $S_y$, and $S_z$—that starts out innocuously enough, and has a nicely-factorable constant term, but is a nightmare in the middle (here covered over with “…”):

\[
0 = 256M^{12} - 2304M^{11} + 32M^{10}(243 + 48S - 68S^2 - 68S^2 - 68S^2 - 68S^2 - 36S_XS_YS_Z) + \ldots \\
+ S(S + S_X^2)(S + S_Y^2)(S + S_Z^2) \\
(S + S_X^2 + S_Y^2 + S_Z^2)(S + S^2 + S^2 + S^2) \\
(S + S_X^2 + S_Y^2 + S_Z^2 - 2S_XS_YS_Z)
\]

Expanding the $S$ terms back into face-area terms doesn’t improve things much. The number of terms in the polynomial grows to over nineteen thousand. However, the constant term breaks down into more factors:

\[
\left(1 - \bar{W}_2^2 - \bar{X}_2^2\right)\left(1 - \bar{W}_2^2 - \bar{Y}_2^2\right)\left(1 - \bar{W}_2^2 - \bar{Z}_2^2\right) \\
\left(1 - \bar{X}_2^2 - \bar{Y}_2^2\right)\left(1 - \bar{Y}_2^2 - \bar{Z}_2^2\right)\left(1 - \bar{X}_2^2 - \bar{Z}_2^2\right) \\
\left(1 - \bar{W}_2^2 - \bar{X}_2^2 - \bar{Y}_2^2 - \bar{Z}_2^2 - 4\bar{W}_2\bar{X}_2\bar{Y}_2\bar{Z}_2\right) \\
\left(1 - \bar{W}_2^2 - \bar{X}_2^2 - \bar{Y}_2^2 - \bar{Z}_2^2 + (\bar{W}_2\bar{X}_2 - \bar{Y}_2\bar{Z}_2)^2\right) \\
\left(1 - \bar{W}_2^2 - \bar{X}_2^2 - \bar{Y}_2^2 - \bar{Z}_2^2 + (\bar{W}_2\bar{Y}_2 - \bar{Z}_2\bar{X}_2)^2\right) \\
\left(1 - \bar{W}_2^2 - \bar{X}_2^2 - \bar{Y}_2^2 - \bar{Z}_2^2 + (\bar{W}_2\bar{Z}_2 - \bar{X}_2\bar{Y}_2)^2\right)
\]

Note the first few factors admit further trigonometric simplification. For instance,

\[
1 - \bar{W}_2^2 - \bar{X}_2^2 = -\frac{1}{2} (\cos W + \cos X) = -\cos \left(\frac{W + X}{2}\right) \cos \left(\frac{W - X}{2}\right)
\]

So far, any further sense in the $M$ polynomial has been elusive.

Nevertheless, converting the $M$ polynomial into, say, an $H_2$ polynomial is a simple as a substitution: replace $M$ with $\bar{H}_2^2 - \bar{H}_2 S_X$. The resulting polynomial has degree 24, but factors into polynomials of degree 12 (with, respectively, 131 and 229 terms).

\[
\left(16\bar{H}_2^{12} - 64\bar{H}_2^{11}S_X + \cdots + S(S + S_X^3)(S + S_Y^2)(S + S_Z^2)\right) \\
\left(16\bar{H}_2^{12} - 128\bar{H}_2^{11}S_X + \cdots + S(S + S_X^3 + S_Y^2)(S + S_X^2 + S_Y^2)(S + S_X^2 + S_Y^2 + S_Z^2 - 2S_XS_YS_Z)\right)
\]
These polynomials, though shorter than the \( M \) polynomial, aren’t any less daunting to study. But, of course, if we seek to tie back into the formula for tetrahedral volume, then we shouldn’t stop at the pseudo-face level; we should proceed to the dihedral angle level. The 2.5th Law of Cosines provides the necessary bridges—for instance, \( \bar{H}_2 = Y_2 \bar{Z}_2 + Y_2 \bar{Z}_2 \bar{A} \)—for converting our polynomial. Writing \( A_{XY} \) for \( Y_2 \bar{Z}_2 \bar{A} \), we have\(^9\)

\[
\begin{align*}
16A_{YZ}^{12} &- 64A_{YZ}^{11} \left( \bar{W}_2 \bar{X}_2 - 2Y_2 \bar{Z}_2 \right) \\
&+ \cdots
\end{align*}
\]

\[
\begin{align*}
+Y_2^2 \bar{Z}_2^2 \left( 1 - \bar{W}_2^2 - \bar{X}_2^2 - \bar{Y}_2^2 + 2\bar{W}_2 \bar{X}_2 \bar{Y}_2 \bar{Z}_2 \right) \\
\left( 1 - \bar{W}_2^2 - \bar{X}_2^2 - \bar{Z}_2^2 + 2\bar{W}_2 \bar{X}_2 \bar{Y}_2 \bar{Z}_2 \right)
\end{align*}
\]

\[
\begin{align*}
\left( \bar{W}_2^2 \bar{Y}_2^2 \bar{Z}_2^2 - (\bar{X}_2 - \bar{W}_2 \bar{Y}_2 \bar{Z}_2)^2 \right) \\
\left( \bar{X}_2^2 \bar{Y}_2^2 \bar{Z}_2^2 - (\bar{W}_2 - \bar{X}_2 \bar{Y}_2 \bar{Z}_2)^2 \right)
\end{align*}
\]

\[
\begin{align*}
16A_{YZ}^{12} &+ 64A_{YZ}^{11} \left( \bar{Y}_2 \bar{Z}_2 - 2\bar{W}_2 \bar{X}_2 \right) \\
&+ \cdots
\end{align*}
\]

\[
\begin{align*}
+\bar{W}_2^2 \bar{X}_2^2 \left( 1 - \bar{W}_2^2 - \bar{Y}_2^2 - \bar{Z}_2^2 - 2\bar{W}_2 \bar{X}_2 \bar{Y}_2 \bar{Z}_2 \right) \\
\left( 1 - \bar{W}_2^2 - \bar{Y}_2^2 - \bar{Z}_2^2 + 2\bar{W}_2 \bar{X}_2 \bar{Y}_2 \bar{Z}_2 \right)
\end{align*}
\]

\[
\begin{align*}
\left( \bar{W}_2^2 \bar{X}_2^2 \bar{Y}_2^2 - (\bar{Z}_2 - \bar{W}_2 \bar{X}_2 \bar{Y}_2)^2 \right) \\
\left( \bar{W}_2^2 \bar{X}_2^2 \bar{Z}_2^2 - (\bar{Y}_2 - \bar{W}_2 \bar{X}_2 \bar{Z}_2)^2 \right)
\end{align*}
\]

Likewise for the other dihedral quantities, \( B_{ZX} := \bar{Z}_2 \bar{X}_2 \bar{B} \), \( C_{XY} := \bar{X}_2 \bar{Y}_2 \bar{C} \), \( D_{WX} := \bar{W}_2 \bar{X}_2 \bar{D} \), \( E_{WY} := \bar{W}_2 \bar{Y}_2 \bar{E} \), and \( F_{WZ} := \bar{W}_2 \bar{Z}_2 \bar{F} \). Currently, we have no information on how many roots of these polynomials may be extraneous, either inherently as individuals (being non-real values or having greater-than-one absolute values), or for failing collectively to satisfy relations such as (4). One should hope for a clear method of weeding out extraneous roots from “appropriate” ones, for otherwise, the face areas of a perfect tetrahedron provide as many as 24 “candidate” measures for each dihedral angle, determining \( 24^6 = 191,102,976 \) different tetrahedra (or \( 24^6/24 = 7,962,624 \), when accounting for tetrahedral symmetries). Nevertheless, we have a foundation for a strategy for computing the volume of our tetrahedron.

**Theorem 4.1 (The Perfect Heron Strategy).** To compute the volume of a perfect hyperbolic tetrahedron from the areas of its faces, use the family of polynomials in \( A_{YZ} \), etc, to determine the candidates for the measures of the dihedral angles \( A, B, C, \) etc. Then substitute the “appropriate” measures into the Derevnin-Mednykh formula.

Not the most satisfying of results.

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\(^9\)Despite appearances in the abbreviated display, the factors do not simply exchange \( Y \) and \( Z \) for \( W \) and \( X \). The first factor, when expanded, has 672 terms; the second, 923.
On a more optimistic note: Perfection provides definite simplification of a tetrahedron’s “hedronometric” Gram matrix representation (although pseudo-face elements remain). Multiplying each factor by one of $\bar{H}_2$, $\bar{J}_2$, and $\bar{K}_2$ — and dividing out to compensate— leaves three equivalent factors.

$$\det G := \frac{-4}{\bar{W}_2 \bar{X}_2 \bar{Y}_2 \bar{Z}_2} \cdot \left( \frac{\bar{J}_2 \bar{K}_2 + \bar{H}_2 - \bar{W}_2 \bar{X}_2 - \bar{Y}_2 \bar{Z}_2}{\bar{H}_2 \bar{J}_2 + \bar{J}_2 - \bar{W}_2 \bar{Y}_2 - \bar{Z}_2 \bar{X}_2} \right)$$

$$= \frac{-4}{\bar{W}_2 \bar{X}_2 \bar{Y}_2 \bar{Z}_2} \cdot \frac{1}{\bar{H}_2 \bar{J}_2} \cdot \left( \frac{\bar{H}_2 \bar{J}_2 \bar{K}_2 + \bar{H}_2^2 - \bar{H}_2 (\bar{W}_2 \bar{X}_2 + \bar{Y}_2 \bar{Z}_2)}{\bar{H}_2 \bar{J}_2 \bar{K}_2 + \bar{J}_2^2 - \bar{J}_2 (\bar{W}_2 \bar{Y}_2 + \bar{Z}_2 \bar{X}_2)} \right)$$

$$= \frac{-4 \left( \bar{H}_2 \bar{J}_2 \bar{K}_2 + M \right)^3}{\bar{W}_2 \bar{X}_2 \bar{Y}_2 \bar{Z}_2 \bar{H}_2 \bar{J}_2 \bar{K}_2}$$

One might wish to take a further step of expressing the perfected $\det G$ in terms of face areas alone; or in terms of pseudo-face areas and $M$ (as these values dominate the above formula). Either way may show $\det G$ to be a more suitable “symmetric” parameter than $M$ (which was defined rather arbitrarily). This author, unfortunately, has yet to find either the patience required for his aging computer to crank through the brute-force manipulations ... or the insight required to side-step them.

### 5. Symmetric Tetrahedra: Using a Custom-Tailored Volume Integral

The article [2] side-steps the Derevnin-Mednykh integral (1), deriving from differential principals an integral tailored to the special case of symmetric hyperbolic tetrahedra, which are defined by having congruent opposing dihedral angles (say, $A$, $B$, and $C$); such tetrahedra have congruent faces with area $W$ satisfying

$$\bar{W} = \frac{\bar{A} + \bar{B} + \bar{C} - 1}{(1 - \bar{A})(1 - \bar{B})(1 - \bar{C})} \sqrt{1 - 2 \bar{A} \bar{B} \bar{C} - \bar{A}^2 - \bar{B}^2 - \bar{C}^2}$$

The new integral is parameterized by three angles ($A$, $B$, $C$) meeting at a vertex (or, equivalently, surrounding a face).

$$V = \int_{v_0}^{\infty} \frac{\arcsin \bar{A}}{\sqrt{v^2 + 1}} + \arcsin \frac{\bar{B}}{\sqrt{v^2 + 1}} + \arcsin \frac{\bar{C}}{\sqrt{v^2 + 1}} - \arcsin \frac{1}{\sqrt{v^2 + 1}} \frac{dv}{v}$$
where

\[ v_0 = \frac{1 - 2\hat{A}\hat{B}\hat{C} - \hat{A}^2 - \hat{B}^2 - \hat{C}^2}{\sqrt{(1 - \hat{A} + \hat{B} + \hat{C}) (1 + \hat{A} - \hat{B} + \hat{C}) (1 + \hat{A} + \hat{B} - \hat{C}) (1 + \hat{A} + \hat{B} + \hat{C})}} \]

This result, being expressed entirely in cosines of the figure’s dihedral angles, has a decidedly more hedronometry-friendly flavor than the Derevnin-Mednykh formula. We can easily use the 2.5th Law of Cosines to re-write the equation in terms of common face area \((W)\) and pseudo-face areas \((H, J, K)\) by making these substitutions:

\[ \tilde{A} = \frac{\hat{H}_2 - \tilde{W}_2^2}{W_2^2}, \quad \tilde{B} = \frac{\hat{J}_2 - \tilde{W}_2^2}{W_2^2}, \quad \tilde{C} = \frac{\hat{K}_2 - \tilde{W}_2^2}{W_2^2} \]

However, this analysis gets more notationally compact if we work in terms of half-angles \((A_2, B_2, C_2)\) and quarter-pseudo-faces \((\hat{H}_4, \hat{J}_4, \hat{K}_4)\) while shifting our trigonometric affections away from cosine and toward sine. We start by noting that (8) implies\(^{10}\)

\[ A_2 = \sqrt{\frac{1 - \tilde{A}}{2}} = \sqrt{\frac{1 - \hat{H}_2}{2W_2^2}}, \quad B_2 = \frac{\hat{J}_4}{W_2}, \quad C_2 = \frac{\hat{K}_4}{W_2} \]

The Third Law of Cosines provides a convenient factorization

\[ 0 = \left( \left( \tilde{W}_2^2 - \hat{H}_4^2 - \hat{J}_4^2 - \hat{K}_4^2 \right) + 2\hat{H}_4\hat{J}_4\hat{K}_4 \right) \left( \left( \tilde{W}_2^2 - \hat{H}_4^2 - \hat{J}_4^2 - \hat{K}_4^2 \right) - 2\hat{H}_4\hat{J}_4\hat{K}_4 \right) \]

that implies

\[ \tilde{W}_2^2 - \hat{H}_4^2 - \hat{J}_4^2 - \hat{K}_4^2 = \pm 2\hat{H}_4\hat{J}_4\hat{K}_4 \]

We deduce that the “±” should be “+”: the product \(2\hat{H}_4\hat{J}_4\hat{K}_4\) is (assumed) non-negative; the left-hand side is equivalent to \(\frac{1}{2}\tilde{W}_2^2 \left( \tilde{A} + \tilde{B} + \tilde{C} - 1 \right)\), which is also non-negative.\(^{11}\) Thus,

\[ \tilde{W}_2^2 = 2\hat{H}_4\hat{J}_4\hat{K}_4 + \hat{H}_4^2 + \hat{J}_4^2 + \hat{K}_4^2 \]

so that the integration endpoint \(v_0\) from (7) becomes

\[ v_0 = \frac{\hat{H}_4\hat{J}_4\hat{K}_4 \left( 1 - 2\hat{H}_4\hat{J}_4\hat{K}_4 - \hat{H}_4^2 - \hat{J}_4^2 - \hat{K}_4^2 \right)}{\left( 2\hat{H}_4\hat{J}_4\hat{K}_4 + \hat{H}_4^2 + \hat{J}_4^2 + \hat{K}_4^2 \right) \sqrt{(\hat{H}_4 + \hat{J}_4\hat{K}_4) (\hat{J}_4 + \hat{H}_4\hat{K}_4) (\hat{K}_4 + \hat{H}_4\hat{J}_4)}} \]

\(^{10}\)Knowing that \(\sin \frac{A}{2}, \sin \frac{B}{2}, \sin \frac{C}{2}\), and \(\sin \frac{W}{2}\) are non-negative —because \(A, B, C,\) and \(W\) are bounded above by \(\pi\) — and taking \(\sin \frac{D}{2}, \sin \frac{F}{2}, \) and \(\sin \frac{G}{2}\) also non-negative on the assumption that \(H, J,\) and \(K\) are bounded above by \(2\pi\).

\(^{11}\)The expression \(\tilde{A} + \tilde{B} + \tilde{C} - 1\) appears in (6) as a factor, among other non-negative factors, of a non-negative quantity.
We can simplify the integral itself ever-so-slightly via the change of variables $t := v\bar{W}_2^2$. With this, we have

**Theorem 5.1 (The Pseudo-Heron Formula for Volume of a Symmetric Hyperbolic Tetrahedron).** A symmetric hyperbolic tetrahedron with pseudo-face areas $H$, $J$, and $K$ (and face area $W$) has volume given by

\[
V = \int_{t_0}^{\infty} \arcsin \frac{W_2^2 - 2H_4^2}{\sqrt{t^2 + W_2^4}} + \arcsin \frac{W_2^2 - 2J_4^2}{\sqrt{t^2 + W_2^4}} + \arcsin \frac{W_2^2 - 2K_4^2}{\sqrt{t^2 + W_2^4}} - \arcsin \frac{\bar{W}_2^2}{\sqrt{t^2 + W_2^4}} \frac{dt}{t}
\]

where

\[
\bar{H}_4 := \sin \frac{H}{4} \quad \bar{J}_4 := \sin \frac{J}{4} \quad \bar{K}_4 := \sin \frac{K}{4}
\]

\[
W_2^2 := \sin^2 \frac{W}{2} = 2\bar{H}_4 \bar{J}_4 \bar{K}_4 + \bar{H}_4^2 + \bar{J}_4^2 + \bar{K}_4^2
\]

\[
t_0 := \frac{\bar{H}_4 \bar{J}_4 \bar{K}_4 \left(1 - 2\bar{H}_4 \bar{J}_4 \bar{K}_4 - \bar{H}_4^2 - \bar{J}_4^2 - \bar{K}_4^2\right)}{\left(\bar{H}_4 + \bar{J}_4 \bar{K}_4\right) \left(\bar{J}_4 + \bar{K}_4 \bar{H}_4\right) \left(\bar{K}_4 + \bar{H}_4 \bar{J}_4\right)}
\]

Given that $\bar{W}_2$ can be expressed in terms of $H_4$, $J_4$, and $K_4$, the formula is effectively “pure pseudo”. In Euclidean space, the corresponding formulas for an “equihedral” tetrahedron are $W = \frac{1}{2} \sqrt{H^2 + J^2 + K^2}$ and $V = \frac{1}{3} \sqrt{HJK}$. (The simplicity of the Euclidean volume result raises the question: Does the hyperbolic volume integral have a simpler representation?)

Let us see how perfection affects this scenario.

A perfect symmetric tetrahedron necessarily has six congruent dihedral angles.\(^\text{12}\) Consequently, its three pseudo-face areas are equal.\(^\text{13}\) This gives us a reduced volume formula of the form

\[
V = \int_{t_0}^{\infty} 3 \arcsin \frac{W_2^2 - 2\bar{H}_4^2}{\sqrt{t^2 + W_2^4}} - \arcsin \frac{\bar{W}_2^2}{\sqrt{t^2 + W_2^4}} \frac{dt}{t}
\]

where

\[
\bar{W}_2^2 = 2\bar{H}_4^3 + 3\bar{H}_4^2
\]

\[
t_0 := \frac{\bar{H}_4^2 \left(1 + \bar{H}_4\right) \left(1 - 2\bar{H}_4\right)}{\sqrt{\bar{H}_4 \left(1 + \bar{H}_4\right)}} = \bar{H}_4 \left(1 - 2\bar{H}_4\right) \sqrt{\bar{H}_4 \left(1 + \bar{H}_4\right)}
\]

The reduced Third Law of Cosines reduces further to provide a simpler bridge between the face area and pseudo-face area.

\[
W_2^2 = 2\bar{H}_4^3 + 3\bar{H}_4^2
\]

We can thus solve for $H$ in terms of $W$, generating three candidates:

\(^\text{12}\) Equality of opposite dihedral angles, coupled with the perfection condition (4), guarantees this.

\(^\text{13}\) Without a geometric interpretation of pseudo-faces, the word “congruent” isn’t appropriate.
\[ \bar{H}_4 = \frac{1}{2} \left( 2 \cos \frac{W - \pi}{3} - 1 \right), \quad \frac{1}{2} \left( 2 \cos \frac{W + \pi}{3} - 1 \right), \quad \frac{1}{2} \left( 1 + 2 \cos \frac{W}{3} \right) \]

The area of the hyperbolic triangle \( W \) is bounded above by \( \pi \), and the area of the pseudo-face \( H \) is bounded above by \( 2\pi \) so that \( \bar{H}_4 \) must be non-negative. This eliminates all candidate values except the first. From this we deduce the following:

**Theorem 5.2 (The Heron Formula for Volume of a Perfect Symmetric Hyperbolic Tetrahedron).**

A perfect symmetric hyperbolic tetrahedron with face area \( W \) (and pseudo-face area \( H \)) has volume given by

\[ V = \int_{t_0}^{\infty} 3 \arcsin \frac{W_2^2 - 2\bar{H}_4^2}{\sqrt{t^2 + W_2^2}} - \arcsin \frac{W_2^2}{\sqrt{t^2 + W_2^2}} \frac{dt}{t} \]

where

\[ W_2 := \sin \frac{W}{2} \quad \bar{H}_4 := \sin \frac{H}{4} = \frac{1}{2} \left( 2 \cos \frac{W - \pi}{3} - 1 \right) \]

and

\[ t_0 = 2 \left( 2 \cos \frac{W - \pi}{3} - 1 \right) \sin^2 \frac{W - \pi}{6} \sqrt{\frac{W}{3} \sin \frac{W + \pi}{3}} \]

Again, the simplicity of the Euclidean counterpart formulas — \( H^2 = \frac{4}{3} W^2 \) and \( V^4 = \frac{64}{2187} W^6 \) — suggests that this result has a more elegant presentation.

**REFERENCES**

http://mathlab.snu.ac.kr/~top/articles/MednykhDerevninHyperbolicTetrahedron.pdf
(See also the Hyperbolic Volume Workshop Homepage: http://mathlab.snu.ac.kr/~top/)
http://mathlab.snu.ac.kr/~top/articles/Tetr_Volume.pdf