HERON-LIKE STRATEGIES FOR NON-EUCLIDEAN TETRAHEDRAL VOLUME

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This author's note "Heron-like Results for Tetrahedral Volume" [1] presents the *Pseudo-Heron Formula* for the volume of a Euclidean tetrahedron in terms of the areas of its four faces and three pseudofaces. It also introduces the *Heron Quartic* that encodes the volume of a "perfect" (or "orthogonal") tetrahedron within a fourth-degree polynomial whose coefficients are expressed in terms of the areas of the four faces alone. This note offers a preliminary reconnaissance exercise in the quest for analogues of these results for a tetrahedron in hyperbolic space.

Briefly, a hyperbolic tetrahedron's volume is a function of the measures of its six dihedral angles. Basic "hedronometry" provides the necessary (and familiar) bridges between these angle measures and face-and-pseudoface areas, guaranteeing a Pseudo-Heron result of some form. Likewise, "perfection" introduces dependencies between faces and pseudofaces, allowing us to eliminate the latter from consideration and compute the volume of perfect tetrahedra in proper Heronic fashion: using their face areas alone.

At least, that's the idea.

Unfortunately, the nature of the key hyperbolic volume function (it's an integral) makes the consequent Pseudo-Heron result less than satisfying, as we cannot give a straightforward explicit (or implicit) formula; and, while perfection does admit elimination of pseudoface elements, the relations involve degree-12 polynomials. At best, our results reduce to strategies on the order of "substitute appropriate solutions of some equations into the established hyperbolic volume formula".

The final section of this note investigates discusses the case of "symmetric" tetrahedra, which admit a volume integral directly parameterized by face and pseudoface areas. This raises the possibility of something similar in the general case.

1. Notation

Denote the faces (and, without confusion, their areas) of a tetrahedron by W, X, Y, Z. Let its pseudofaces (and their areas) be H, J, K. Finally, name the dihedral angles (and their measures) as follows, where $\angle PQ$ indicates the angle between faces P and Q:

$$A := \angle YZ \qquad B := \angle ZX \qquad C := \angle XY$$
$$D := \angle WX \qquad E := \angle WY \qquad F := \angle WZ$$

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To conserve space in long formulas, we employ the following "Morse Code" for cosines and sines of areas or angles:

$$\ddot{X} := \cos X \text{ or } \cosh X \qquad \overline{X} := \sin X \text{ or } \sinh X$$

where context will make clear whether circular or hyperbolic functions are intended. Moreover, for readability, we write "X_2" for "$X/2$". Thus, $\ddot{X}_2 := \cos\frac{1}{2}X$.

2. Two Hedronometric Formulas

We pull two key results from a previous note [3].

The first result forms the bridge between dihedral angle information and face-and-pseudoface area information (and it serves to formally define the pseudo-elements).[1]

Theorem 1 (Law of Opposite Cosines, with formal definition of Pseudofaces)**.**

$$\text{(LoOC)} \qquad \begin{aligned} \ddot{Y}_2\ddot{Z}_2 + \overline{Y_2}\,\overline{Z_2}\ddot{A} \;&=\; \ddot{H}_2 \;=\; \ddot{W}_2\ddot{X}_2 + \overline{W_2}\,\overline{X_2}\ddot{D} \\ \ddot{Z}_2\ddot{X}_2 + \overline{Z_2}\,\overline{X_2}\ddot{B} \;&=\; \ddot{J}_2 \;=\; \ddot{W}_2\ddot{Y}_2 + \overline{W_2}\,\overline{Y_2}\ddot{E} \\ \ddot{X}_2\ddot{Y}_2 + \overline{X_2}\,\overline{Y_2}\ddot{C} \;&=\; \ddot{K}_2 \;=\; \ddot{W}_2\ddot{Z}_2 + \overline{W_2}\,\overline{Z_2}\ddot{F} \end{aligned}$$

We introduce the convention that each of H, J, K is bounded by 2π, although this consideration does not enter into discussion.[2]

The second key result is described as a "symmetric, face-agnostic" Law of Cosines.[3]

Theorem 2 (Law of Face Cosines)**.**

$$\text{(LoFC)} \qquad \begin{aligned} 0 \;=\; & 1 - \ddot{W}_2^{\,2} - \ddot{X}_2^{\,2} - \ddot{Y}_2^{\,2} - \ddot{Z}_2^{\,2} - 4\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2 \\ & - \ddot{H}_2^{\,2} - \ddot{J}_2^{\,2} - \ddot{K}_2^{\,2} - 2\ddot{H}_2\ddot{J}_2\ddot{K}_2 \\ & + 2\ddot{H}_2(\ddot{W}_2\ddot{X}_2 + \ddot{Y}_2\ddot{Z}_2) + 2\ddot{J}_2(\ddot{W}_2\ddot{Y}_2 + \ddot{Z}_2\ddot{X}_2) + 2\ddot{K}_2(\ddot{W}_2\ddot{Z}_2 + \ddot{X}_2\ddot{Y}_2) \end{aligned}$$

3. The Volume Formula, and the Pseudo-Heron Strategy

Derevnin and Mednykh [4] offer this symmetric formula for the volume of a (compact) tetrahedron with dihedral angles A, B, C, D, E, F:

$$\text{(1)} \qquad V = -\frac{1}{4}\int_{\theta_0-\phi}^{\theta_0+\phi} \frac{\cos\frac{A+B+C+\theta}{2}\cos\frac{A+E+F+\theta}{2}\cos\frac{D+B+F+\theta}{2}\cos\frac{D+E+C+\theta}{2}}{\sin\frac{A+D+B+E+\theta}{2}\sin\frac{B+E+C+F+\theta}{2}\sin\frac{C+F+A+D+\theta}{2}\sin\frac{\theta}{2}}\,d\theta$$

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[1] The Euclidean counterparts are $Y^2 + Z^2 - 2YZ\cos A = H^2 = W^2 + X^2 - 2WX\cos D$, etc.

[2] Observe that, as dihedral angle A ranges from 0 to π, the value of the expression for $\cos\frac{1}{2}H$ ranges from $\cos\frac{1}{2}(Y-Z)$ to $\cos\frac{1}{2}(Y+Z)$. It is not unreasonable, then, to declare that H falls between $Y-Z$ and $Y+Z$; under this agreement, the fact that hyperbolic triangles have a maximum area of π provides an absolute maximum value of 2π on H. Consequently, pseudofaces are conceptual *quadrilaterals*. This is consistent with the fact, as described in [2], that the pseudofaces of a Euclidean tetrahedron are the quadrilateral projections of the figure into planes parallel to pairs of opposite edges. That said, hyperbolic pseudofaces so far lack such a specific geometric interpretation.

[3] The Euclidean counterpart has no cosines at all: $W^2 + X^2 + Y^2 + Z^2 = H^2 + J^2 + K^2$.

where

$$\theta_0 := \tan^{-1}\frac{k_2}{k_1} \qquad\qquad \phi := \tan^{-1}\frac{k_4}{k_3}$$

and

$$\begin{aligned}
k_1 :=& -\cos(A+B+C+D+E+F) - \cos(A+D) - \cos(B+E) - \cos(C+F) \\
&- \cos(D+E+F) - \cos(D+B+C) - \cos(A+E+C) - \cos(A+B+F) \\
k_2 :=& \quad \sin(A+B+C+D+E+F) + \sin(A+D) + \sin(B+E) + \sin(C+F) \\
&+ \sin(D+E+F) + \sin(D+B+C) + \sin(A+E+C) + \sin(A+B+F) \\
k_3 :=& \quad 2\,(\ \sin A \sin D + \sin B \sin E + \sin C \sin F\ ) \\
k_4 :=& \quad \sqrt{k_1^2 + k_2^2 - k_3^2}
\end{aligned}$$

Importantly, k_4 is closely tied to the tetrahedron's Gram matrix, G:

$$-\tfrac{1}{4}k_4^2 \ = \ \det G \ := \ \det \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix}$$

Note that $\det G$ (and k_4) is non-zero for a non-degenerate tetrahedron.[4]

Now, the Derevnin-Mednykh integral, coupled with the Law of Opposite Cosines, amounts to a Pseudo-Heron Formula, as we have all angle references converted into expressions involving areas of faces and pseudofaces.

Unfortunately, simply "plugging in" doesn't yield particularly attractive results. When expanded into trig functions of individual dihedral angles, the expressions for k_1, k_2, k_3 involve not just cosines, but (un-squared) sines; at best, we can express these using square roots, as in

$$\begin{aligned}
\sin A &= \frac{1}{\overline{Y_2}\,\overline{Z_2}}\sqrt{1 + 2\ddot{H}_2\ddot{Y}_2\ddot{Z}_2 - \ddot{H}_2{}^2 - \ddot{Y}_2{}^2 - \ddot{Z}_2{}^2} \\
&= \frac{2}{\overline{Y_2}\,\overline{Z_2}}\sqrt{\sin\tfrac{1}{4}(H+Y+Z)\sin\tfrac{1}{4}(-H+Y+Z)\sin\tfrac{1}{4}(H-Y+Z)\sin\tfrac{1}{4}(H+Y-Z)}
\end{aligned}$$

There seems to be no benefit, then, in expanding ks into face-and-pseudoface form; they just get longer, and they don't make the integral any easier, so they don't get us any closer to an explicit

[4]If $k_4 = 0$, then either $\phi = 0$ or else ϕ involves an indeterminate fraction that requires $k_3 = 0$. In the first case, the volume integral's limits match, so that $V = 0$; in the second case, we must have at least three dihedral angles of measure 0.

formula for volume. Contrast this with the Euclidean case, for which volume is related to a simple polynomial in the seven areas.[5] Nevertheless, we can articulate a rather obvious make-do *strategy*:

Theorem 3 (The Pseudo-Heron Strategy)**.** *To compute the volume of a hyperbolic tetrahedron from the areas of its faces and pseudofaces,*

(1) *Substitute into (LoOC) to determine dihedral angles A, B, C, D, E, F.*
(2) *Invoke the Derevnin-Mednykh integral.*

We'll close this section with the observations that the "all cosines" nature of $\det G$ leads to a not-unpleasant face-and-pseudoface form (upon reducing *modulo* the Law of Face Cosines):

$$\det G = -4\,\frac{\langle H_2\rangle\langle J_2\rangle\langle K_2\rangle}{\overline{W_2}^2\,\overline{X_2}^2\,\overline{Y_2}^2\,\overline{Z_2}^2}$$

where[6]

$$\begin{aligned}
\langle H_2\rangle &:= \ddot{H}_2 + \ddot{J}_2\ddot{K}_2 - \ddot{W}_2\ddot{X}_2 - \ddot{Y}_2\ddot{Z}_2\\
\langle J_2\rangle &:= \ddot{J}_2 + \ddot{K}_2\ddot{H}_2 - \ddot{W}_2\ddot{Y}_2 - \ddot{Z}_2\ddot{X}_2\\
\langle K_2\rangle &:= \ddot{K}_2 + \ddot{H}_2\ddot{J}_2 - \ddot{W}_2\ddot{Z}_2 - \ddot{X}_2\ddot{Y}_2
\end{aligned}$$

4. When Perfection isn't Perfect

A *perfect* tetrahedron[7] is one in which each pair of opposite edges are orthogonal. The family of perfect tetrahedra includes right-corner tetrahedra (with three orthogonal edges meeting at a vertex) and regular tetrahedra (with equilateral faces). Perfection induces geometric dependencies that reduce the degrees of freedom in such a tetrahedron from six to four, so that face areas alone may characterize the figure. In Euclidean space, this leads to the Heron Quartic (see "Heron-like Results for Tetrahedral Volume"), a polynomial that allows us to compute the figure's volume from its face areas; in hyperbolic space, perfection gets us as far as a 12th-degree polynomial with who-knows-how-many extraneous roots. Perfection does have a simplifying effect on the Gram matrix, however. We will review some salient properties of perfect tetrahedra here.

As it happens, hyperbolic and Euclidean hedronometry share a common characterization of perfection, namely:

$$\cos A\cos D = \cos B\cos E = \cos C\cos F \tag{2}$$

[5] As in

$$\begin{aligned}
81V^4 &= H^2J^2K^2 - 2(WX-YZ)(WY-ZX)(WZ-XY)\\
&\quad - H^2(WX-YZ)^2 - J^2(WY-ZX)^2 - K^2(WZ-XY)^2
\end{aligned}$$

[6] 2022 update: The expressions $\langle H_2\rangle$, $\langle J_2\rangle$, $\langle K_2\rangle$ play a large role in expressions for various other metric properties. See this author's later note, "Hedronometric Formulas for a Hyperbolic Tetrahedron".

[7] "Perfect" is this author's term. In Euclidean geometry, perfect tetrahedra are known as *orthogonal*, for having pairs of orthogonal opposite edges, or *orthocentric*, for having concurrent altitudes; the properties are equivalent.

From the Law of Opposite Cosines, we deduce

$$\ddot{H}_2\langle H_2\rangle = \ddot{J}_2\langle J_2\rangle = \ddot{K}_2\langle K_2\rangle \tag{3}$$

In "Heron-like Results" [1], the Euclidean counterpart of this relation gives rise to the *Heron Quartic* relating volume to face-areas alone. As mentioned, a similar result in hyperbolic space is unknown. Short of an explicit volume formula, we can outline a strategy for determining a perfect tetrahedron's dihedral angles in terms of its face-areas by adding a step to the Pseudo-Heron Strategy:

Theorem 4 (The Perfect Heron Strategy)**.** *To compute the volume of a perfect hyperbolic tetrahedron from the areas of its faces,*

1. *Solve the non-linear system of (3) and (LoFC) for* $\ddot{H}_2$*,* $\ddot{J}_2$*,* $\ddot{K}_2$*.*
2. *Substitute into (LoOC) to determine dihedral angles* A*,* B*,* C*,* D*,* E*,* F*.*
3. *Invoke the Derevnin-Mednykh integral.*

Of course, step (1) here is more easily said (or typed) than done. Applying, say, the method of resultants to the system yields degree-12 polynomials for $\ddot{H}_2$, $\ddot{J}_2$, $\ddot{K}_2$. Currently, we have no information on how many roots of these polynomials may be extraneous, either inherently as individuals (being non-real values or having absolute values greater than one), or for failing collectively to determine viable sets of pseudoface areas.

Symbolically, the situation is a bit of a mess; for example, the $\ddot{H}_2$ polynomial has the form

$$\begin{aligned} 0 = 16\ddot{H}_2{}^{12} &- 64\ddot{H}_2{}^{11}S_X + 24\ddot{H}_2{}^{10}(4S_X^2 - 3) - 8\ddot{H}_2{}^{9}(8S_X^3 - 42S_X - 7S_YS_Z) \\ &+ \ddot{H}_2{}^{8}(16S_X^4 - 176S_XS_YS_Z - 584S_X^2 - 68S_Y^2 - 68S_Z^2 + 48S + 81) \\ &+ \cdots + S(S + S_Y^2)(S + S_Z^2)(S + S_Y^2 + S_Z^2) \end{aligned}$$

where

$$S_X := \ddot{W}_2\ddot{X}_2 + \ddot{Y}_2\ddot{Z}_2 \qquad S_Y := \ddot{W}_2\ddot{Y}_2 + \ddot{Z}_2\ddot{X}_2 \qquad S_Z := \ddot{W}_2\ddot{Z}_2 + \ddot{X}_2\ddot{Y}_2$$

$$S := 1 - \ddot{W}_2{}^2 - \ddot{X}_2{}^2 - \ddot{Y}_2{}^2 - \ddot{Z}_2{}^2 - 4\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2$$

Trigonometry doesn't seem to help clear things up, although the last factor of the constant term simplifies:

$$S + S_Y^2 + S_Z^2 = \cos(W_2 - X_2)\cos(W_2 + X_2)\cos(Y_2 - Z_2)\cos(Y_2 + Z_2)$$

Note. Defining $M := \ddot{H}_2\langle H_2\rangle = \ddot{J}_2\langle J_2\rangle = \ddot{K}_2\langle K_2\rangle$, one can eliminate all pseudoface elements from the system (3) and (LoFC) to obtain a degree-12 polynomial in M. It's much longer than the polynomial for $\ddot{H}_2$, and no less inscrutable, but it has the advantage of symmetry with respect to the face areas. The value M is notable, because we can write a perfect hyperbolic tetrahedron's Gram determinant as

$$\det G = \frac{-4M^3}{\overline{W_2}^2\,\overline{X_2}^2\,\overline{Y_2}^2\,\overline{Z_2}^2 \cdot \ddot{H}_2\ddot{J}_2\ddot{K}_2}$$

One might wish to take a further step of expressing the perfected $\det G$ in terms of face areas alone; or in terms of pseudoface areas and M. Either way may show $\det G$ to be a more suitable "symmetric" parameter than M. This author, unfortunately, has yet to find either the patience required to crank through the brute-force manipulations or the insight required to side-step them.

5. Symmetric Tetrahedra: Using a Custom-Tailored Volume Integral

The article [5] derives a volume integral tailored to the special case of a *symmetric* hyperbolic tetrahedron, which is defined by having congruent opposing dihedral angles ($A = D$, $B = E$, $C = F$). The integral is parameterized by the three angles thusly:

$$V = \int_{v_0}^{\infty} \arcsin \frac{\ddot{A}}{\sqrt{v^2+1}} + \arcsin \frac{\ddot{B}}{\sqrt{v^2+1}} + \arcsin \frac{\ddot{C}}{\sqrt{v^2+1}} - \arcsin \frac{1}{\sqrt{v^2+1}} \; \frac{dv}{v} \tag{4}$$

where

$$v_0 := \frac{1 - 2\ddot{A}\ddot{B}\ddot{C} - \ddot{A}^2 - \ddot{B}^2 - \ddot{C}^2}{\sqrt{(-1+\ddot{A}+\ddot{B}+\ddot{C})(1-\ddot{A}+\ddot{B}+\ddot{C})(1+\ddot{A}-\ddot{B}+\ddot{C})(1+\ddot{A}+\ddot{B}-\ddot{C})}}$$

This result, being expressed entirely in cosines of the figure's dihedral angles, has a decidedly more hedronometry-friendly flavor than the Derevnin-Mednykh formula. We can easily re-write the integral in terms of common face area (W) and pseudoface areas (H, J, K).

First, note that W, calculated by its angular defect converted to dihedral angles, satisfies

$$\overline{W_2} = \frac{1 - \overline{A_2}^2 - \overline{B_2}^2 - \overline{C_2}^2}{2\overline{A_2}\,\overline{B_2}\,\overline{C_2}}$$

Also, by the Law of Opposite Cosines,

$$\ddot{A} = \frac{\overline{W_2}^2 - 2\overline{H_4}^2}{\overline{W_2}^2} \qquad \ddot{B} = \frac{\overline{W_2}^2 - 2\overline{J_4}^2}{\overline{W_2}^2} \qquad \ddot{C} = \frac{\overline{W_2}^2 - 2\overline{K_4}^2}{\overline{W_2}^2} \tag{5}$$

so that

$$\overline{A_2} = \frac{\overline{H_4}}{\overline{W_2}} \qquad \overline{B_2} = \frac{\overline{J_4}}{\overline{W_2}} \qquad \overline{C_2} = \frac{\overline{K_4}}{\overline{W_2}} \qquad \to \qquad \overline{W_2}^2 = 2\overline{H_4}\,\overline{J_4}\,\overline{K_4} + \overline{H_4}^2 + \overline{J_4}^2 + \overline{K_4}^2 \tag{6}$$

whence

$$v_0 = \frac{\overline{H_4}\,\overline{J_4}\,\overline{K_4}\left(1 - \overline{W_2}^2\right)}{\overline{W_2}^2\sqrt{(\overline{H_4} + \overline{J_4}\,\overline{K_4})(\overline{J_4} + \overline{K_4}\,\overline{H_4})(\overline{K_4} + \overline{H_4}\,\overline{J_4})}}$$

We can simplify the integral itself ever-so-slightly via the change of variables $t := v\overline{W_2}^2$. With this, we have

Theorem 5 (A Pseudo-Heron Formula for the Volume of a Symmetric Hyperbolic Tetrahedron)**.** *A symmetric hyperbolic tetrahedron with pseudoface areas H, J, K has volume given by*

$$V = \int_{t_0}^{\infty} \arcsin \frac{T - 2\overline{H_4}^2}{\sqrt{t^2+T^2}} + \arcsin \frac{T - 2\overline{J_4}^2}{\sqrt{t^2+T^2}} + \arcsin \frac{T - 2\overline{K_4}^2}{\sqrt{t^2+T^2}} - \arcsin \frac{T}{\sqrt{t^2+T^2}} \; \frac{dt}{t}$$

where

$$t_0 := \frac{\overline{H_4}\,\overline{J_4}\,\overline{K_4}\,(1-T)}{\sqrt{(\overline{H_4}+\overline{J_4}\,\overline{K_4})(\overline{J_4}+\overline{K_4}\,\overline{H_4})(\overline{K_4}+\overline{H_4}\,\overline{J_4})}} \qquad T := 2\overline{H_4}\,\overline{J_4}\,\overline{K_4} + \overline{H_4}^2 + \overline{J_4}^2 + \overline{K_4}^2 \left(= \overline{W_2}^2\right)$$

Observe that the formula is "pure pseudo". In Euclidean space, the counterpart formula for the volume of an "equihedral" tetrahedron is given by $9V^2 = HJK$.

5.1. **Perfect Symmetric Tetrahedra.** By (2), the dihedral angles of a perfect symmetric tetrahedron satisfy $\cos^2 A = \cos^2 B = \cos^2 C$. We can deduce that *all dihedral angles of a perfect symmetric tetrahedron are equal*,[8] and thus also all pseudoface areas. We have

Theorem 6 (A Pseudo-Heron Formula for the Volume of a Perfect Symmetric Hyperbolic Tetrahedron)**.** *A non-degenerate perfect symmetric hyperbolic tetrahedron with pseudofaces of area H has volume given by*

$$V = \int_{t_0}^{\infty} 3\arcsin\frac{T - 2\overline{H_4}^2}{\sqrt{t^2+T^2}} - \arcsin\frac{T}{\sqrt{t^2+T^2}}\;\frac{dt}{t}$$

where

$$t_0 := \overline{H_4}(1-2\overline{H_4})\sqrt{\overline{H_4}(1+\overline{H_4})} \qquad T := \overline{H_4}^2(3+2\overline{H_4})\ \left(= \overline{W_2}^2\right)$$

A "standard"-face formulation arises by solving the cubic $\overline{W_2}^2 = \overline{H_4}^2(3+2\overline{H_4})$ for $\overline{H_4}$. We get three candidates:

$$2\sin W_6\cos(W_6+\pi_6) \qquad -2\sin W_6\cos(W_6-\pi_6) \qquad -2\cos(W_6+\pi_6)\cos(W_6-\pi_6)$$

Since W is bounded above by π, and since $\overline{H_4}$ must be non-negative, only the first option is viable. Thus, we have

Theorem 7 (A Heron Formula for the Volume of a Perfect Symmetric Hyperbolic Tetrahedron)**.** *A non-degenerate perfect symmetric hyperbolic tetrahedron with faces of area W has volume given by*

$$V = \int_{t_0}^{\infty} 3\arcsin\frac{\overline{W_2}^2 - S}{\sqrt{t^2+\overline{W_2}^4}} - \arcsin\frac{\overline{W_2}^2}{\sqrt{t^2+\overline{W_2}^4}}\;\frac{dt}{t}$$

where

$$t_0 := 8\sin W_6\cos(W_6+\pi_6)\sin^2(W_6-\pi_6)\sqrt{\sin W_3\sin(W_3+\pi_3)}$$

$$S := 8\sin^2 W_6\cos^2(W_6+\pi_6)\ \left(= 2\overline{H}_4^2\right)$$

The Euclidean counterpart of this formula is $3^7V^4 = 2^6W^6$.

[8] At least two angles are equal; say, $A = B$. Then C is either equal to A, or supplementary to A. In the latter case, $\overline{W_2}$ reduces to $-\frac{1}{2}\sec A_2$, but $\overline{W_2}$ cannot be negative.

References

[1] Blue. *Heron-Like Results for Tetrahedral Volume*. 2005. URL: `http://tricochet.com/mathdocs/heron4tet.pdf`.

[2] Blue. *Pseudofaces of Tetrahedra*. 2005. URL: `http://tricochet.com/mathdocs/pseudofaces.pdf`.

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[4] D. A. Derevnin and A. D. Mednykh. "A formula for the volume of a hyperbolic tetrahedron". In: *Communications of the Moscow Mathematical Society* 60.2 (2005), pp. 346–348.

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