HERON-LIKE STRATEGIES FOR NON-EUCLIDEAN TETRAHEDRAL VOLUME

B.D.S. "DON" MCCONNELL

The note [3] presents the *Pseudo-Heron Formula* for the volume of a (Euclidean) tetrahedron in terms of the areas of its four faces and three *pseudo-faces*. It also introduces the *Heron Quartic* that encodes the volume of a "perfect" tetrahedron within a fourth-degree polynomial whose coefficients are expressed in terms of the areas of the four faces. This note offers a preliminary reconnaissance exercise in the quest for analogues of these results for tetrahedra in hyperbolic space.

Briefly, a hyperbolic tetrahedron's volume is a function of the measures of its six dihedral angles. Basic "hedronometry" provides the necessary (and familiar) bridges between these angle measures and face-and-pseudo-face areas, guaranteeing a Pseudo-Heron result of some form. Likewise, "perfection" introduces dependencies between (proper) faces and pseudo-faces, allowing us to eliminate the latter from consideration and compute the volume of perfect tetrahedra in proper Heronic fashion: using their face areas alone.

At least, that's the idea.

Unfortunately, the nature of the key hyperbolic volume function (it's an integral) makes the consequent Pseudo-Heron result less than satisfying, as we cannot give a straightforward explicit (or implicit) formula; and, while perfection does admit elimination of pseudo-face elements, the relations involve 24th-degree polynomials. At best, our results are reduced to *strategies* on the order of "substitute appropriate solutions of some equations into the established hyperbolic volume formula".

The final section of this note investigates the possibility of deriving a custom integral for at least the Pseudo-Heron case, though the author has not yet engaged in any significant investigation of that approach.

1. Notation

Denote the faces (and, without confusion, their areas) of a tetrahedron by W, X, Y, and Z; let its pseudo-faces (and areas) be H, J, and K. Finally, name the dihedral angles (and their measures)

$$\begin{array}{ccc} A := \angle YZ & B := \angle ZX & C := \angle XY \\ D := \angle WX & E := \angle WY & F := \angle WZ \end{array}$$

Date: 1 March, 2009.

To conserve space in long formulas, we employ the following "Morse Code" for cosines and sines of areas or angles:

$$\ddot{X} := \cos X \qquad \bar{X} := \sin X$$

Moreover, for typographical convenience, as many formulas involving face areas feature trigonometric functions of *half*-areas, we write, eg, "X_2" for "$\frac{X}{2}$". Thus, $\ddot{X}_2 := \cos\frac{X}{2}$.

2. Two Hedronometric Formulas

We pull two results from [4] for reference purposes.

The first result forms the bridge between dihedral angle information and face-and-pseudo-face area information (and it serves to formally define the pseudo elements). We introduce the convention that H, J, and K be bounded by 2π, although this consideration does not enter into discussion until Section 5, where it is invoked to settle a sign ambiguity.[1]

Theorem 2.1 (The Second-and-a-Halfth ("2.5th") Law of Cosines)**.**

$$\begin{aligned}
\ddot{Y}_2\ddot{Z}_2 + \bar{Y}_2\bar{Z}_2\ddot{A} &= \ddot{H}_2 = \ddot{W}_2\ddot{X}_2 + \bar{W}_2\bar{X}_2\ddot{D} \\
\ddot{Z}_2\ddot{X}_2 + \bar{Z}_2\bar{X}_2\ddot{B} &= \ddot{J}_2 = \ddot{W}_2\ddot{Y}_2 + \bar{W}_2\bar{Y}_2\ddot{E} \\
\ddot{X}_2\ddot{Y}_2 + \bar{X}_2\bar{Y}_2\ddot{C} &= \ddot{K}_2 = \ddot{W}_2\ddot{Z}_2 + \bar{W}_2\bar{Z}_2\ddot{F}
\end{aligned}$$

The second result is described as a "symmetric, face-agnostic" Law of Cosines, which here is dubbed the Third Law[2].

[1]As described in [5], the pseudo-faces of a *Euclidean* tetrahedron are the projections of the figure into planes parallel to pairs of opposite edges. No geometric interpretation of non-Euclidean pseudo-faces is (yet) known; they merely serve to make the 2.5th Law of Cosines resemble the laws of cosines relating edges and angles of hyperbolic triangles. (As a point of history, Euclidean pseudo-faces first appeared as formal definitions in a Law of Cosines.) Observe that as, for instance, dihedral angle A ranges from 0 to π, the value of the expression for $\cos\frac{H}{2}$ ranges from $\cos\frac{Y-Z}{2}$ to $\cos\frac{Y+Z}{2}$. It is not unreasonable, then, to declare that H falls between $Y-Z$ and $Y+Z$; under this agreement, the fact that hyperbolic triangles have a maximum area of π provides an absolute maximum value of 2π on H.

[2]This is, in fact, an analogue of the Sum of Squares formula for Euclidean tetrahedra (from [3]): with each cosine term expanded as a power series, and then with all but the lowest-power —in this case, fourth-power— terms discarded, the Third Law reduces to

$$0 \approx \frac{1}{64}\left(W^2 + X^2 + Y^2 + Z^2 - H^2 - J^2 - K^2\right)^2$$

Thus, as a tetrahedron approaches infinitesimal size, the relation among its face areas approaches the Euclidean result, $W^2 + X^2 + Y^2 + Z^2 = H^2 + J^2 + K^2$.

Theorem 2.2 (The Third Law of Cosines)**.**

$$\begin{aligned}0 \;=\;& 1-\ddot{W}_2^{\,2}-\ddot{X}_2^{\,2}-\ddot{Y}_2^{\,2}-\ddot{Z}_2^{\,2}-4\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2\\ &-\ddot{H}_2^{\,2}-\ddot{J}_2^{\,2}-\ddot{K}_2^{\,2}-2\ddot{H}_2\ddot{J}_2\ddot{K}_2\\ &+2\ddot{H}_2\left(\ddot{W}_2\ddot{X}_2+\ddot{Y}_2\ddot{Z}_2\right)+2\ddot{J}_2\left(\ddot{W}_2\ddot{Y}_2+\ddot{Z}_2\ddot{X}_2\right)+2\ddot{K}_2\left(\ddot{W}_2\ddot{Z}_2+\ddot{X}_2\ddot{Y}_2\right)\end{aligned}$$

3. The Volume Formula, and the Pseudo-Heron Strategy

Derevnin and Mednykh [1] offer this formula for the volume of a (compact) tetrahedron with dihedral angles A, B, C, D, E, F:

$$V=-\frac{1}{4}\int_{\theta_0-\phi}^{\theta_0+\phi}\log\frac{\cos\frac{A+B+C+\theta}{2}\cos\frac{A+E+F+\theta}{2}\cos\frac{D+B+F+\theta}{2}\cos\frac{D+E+C+\theta}{2}}{\sin\frac{A+D+B+E+\theta}{2}\sin\frac{A+D+C+F+\theta}{2}\sin\frac{B+E+C+F+\theta}{2}\sin\frac{\theta}{2}}d\theta \tag{1}$$

where

$$\theta_0:=\tan^{-1}\frac{k_2}{k_1}\qquad \phi:=\tan^{-1}\frac{k_4}{k_3}$$

and

$$\begin{aligned}k_1 \;:=\;& -\left(\cos(A+B+C+D+E+F)+\cos(A+D)+\cos(B+E)+\cos(C+F)\right.\\ &\left.+\cos(D+E+F)+\cos(D+B+C)+\cos(A+E+C)+\cos(A+B+F)\right)\end{aligned}$$

$$\begin{aligned}k_2 \;:=\;& \sin(A+B+C+D+E+F)+\sin(A+D)+\sin(B+E)+\sin(C+F)\\ &+\sin(D+E+F)+\sin(D+B+C)+\sin(A+E+C)+\sin(A+B+F)\end{aligned}$$

$$k_3 \;:=\; 2\left(\sin A\sin D+\sin B\sin E+\sin C\sin F\right)$$

$$k_4 \;:=\; \sqrt{k_1^2+k_2^2-k_3^2}$$

Importantly, the k_4 value is closely tied to the tetrahedron's Gram matrix, G:

$$k_4^2=-4\det\begin{pmatrix}1 & -\cos A & -\cos B & -\cos F\\ -\cos A & 1 & -\cos C & -\cos E\\ -\cos B & -\cos C & 1 & -\cos D\\ -\cos F & -\cos E & -\cos D & 1\end{pmatrix}=-4\det G$$

Note that $\det G$ (and k_4) is non-zero for non-degenerate tetrahedra.[3]

[3]If $k_4=0$, then $\phi=0$ or ϕ involves an indeterminate fraction requiring $k_3=0$. In the first case, the limits of integration in the volume forumla match, yielding $V=0$. In the second case, we must have at least three angles of measure 0.

Now, the Derevnin-Mednykh integral, coupled with the 2.5th Law of Cosines, amounts to a Pseudo-Heron Formula, as we have all angle references converted into expressions involving areas of faces and pseudo-faces.

Unfortunately, just "plugging in" doesn't yield attractive results. When expanded into trig functions of individual dihedral angles, the expressions for k_1, k_2, and k_3 involve not just cosines, but (un-squared) *sines*; at best, the 2.5th Law would have us express these using square roots, as in

$$\sin A = \frac{\sqrt{1 + 2\ddot{H}_2\ddot{Y}_2\ddot{Z}_2 - \ddot{H}_2^{\,2} - \ddot{Y}_2^{\,2} - \ddot{Z}_2^{\,2}}}{\bar{Y}_2\bar{Z}_2}$$

There seems to be little benefit, then, in converting the ks into face-and-pseudo-face form. Better to treat this as a *strategy*, instead of a formula:

Theorem 3.1 (The Pseudo-Heron Strategy)**.** *To compute the volume of a hyperbolic tetrahedron from the areas of its faces and pseudo-faces, use the* 2.5*th Law of Cosines to determine the measures of the dihedral angles A, B, C, etc. (via their cosines), then substitute these measures into the Derevnin-Mednykh formula.*

We'll close this section with the observations that the "all cosines" nature of k_4 leads to a not-unpleasant face-and-pseudo-face form:[4]

$$k_4 = \frac{4}{\bar{W}_2\bar{X}_2\bar{Y}_2\bar{Z}_2} \cdot \sqrt{\begin{array}{l} \left(\ddot{H}_2 + \ddot{J}_2\ddot{K}_2 - \ddot{W}_2\ddot{X}_2 - \ddot{Y}_2\ddot{Z}_2\right) \\ \left(\ddot{J}_2 + \ddot{K}_2\ddot{H}_2 - \ddot{W}_2\ddot{Y}_2 - \ddot{Z}_2\ddot{X}_2\right) \\ \left(\ddot{K}_2 + \ddot{H}_2\ddot{J}_2 - \ddot{W}_2\ddot{Z}_2 - \ddot{X}_2\ddot{Y}_2\right) \end{array}}$$

Perhaps the lesson of k_4 is that the whole of the "Pseudo-Heron Strategy" somehow admits reduction to a more compact form. Or perhaps not.

4. When Perfection Isn't Perfect

A *perfect* tetrahedron[5] is one in which each pair of opposite edges is orthogonal. The family of perfect tetrahedra includes right-corner tetrahedra (with three orthogonal edges meeting at a vertex) and regular tetrahedra (with equilateral faces). Perfection induces geometric dependencies that reduce the degrees of freedom in such a tetrahedron from six to four, so that face areas alone may characterize the figure. In Euclidean space, this leads to the *Heron Quartic* (see [3]), a polynomial that allows us to compute the figure's volume from its face areas; in hyperbolic space, perfection gets us as far as a 12th-degree polynomial with who-knows-how-many extraneous roots. Perfection *does* have an simplifying effect on the Gram matrix, however. We will review some salient properties of perfect tetrahedra here.

[4]This form does not arise simply from a straight substitution of the dihedral angle cosines with their face-and-pseudo-face equivalents. Doing *that* yields an unfactorable 96-term expression. The compact form comes from reducing the expansion *modulo* the Third Law of Cosines.

[5]Also known as an *orthogonal* tetrahedron.

Let α, β, and γ be the face-angles —in X, Y, and Z, respectively— having a common vertex, O. Let $a := OP$, $b := OQ$, and $c := OR$ be edges, with a opposite α, etc. We may assume that α is not a right angle. Therefore, the perpendicular from R meets OQ at a point $T \neq O$. Perfection requires that $RTP \perp OQ$, which implies that T is also the foot of the perpendicular from P to OQ. By the trigonometry of hyperbolic triangles,

$$\tanh |OT| = \tanh c \cos\alpha = \tanh a \cos\gamma$$

so that $\cos\alpha / \tanh a = \cos\gamma / \tanh c$. Symmetry provides that we have a value m such that[6]

$$\frac{\cos\alpha}{\tanh a} = \frac{\cos\beta}{\tanh b} = \frac{\cos\gamma}{\tanh c} = \frac{1}{m} \tag{2}$$

Writing $\ddot{x}$ for $\cosh x$ and $\bar{x}$ for $\sinh x$, we incorporate (2) into the hyperbolic Law of Cosines formula for $d := QR$.

$$\ddot{d} = \ddot{b}\ddot{c} - \bar{b}\bar{c}\ddot{\alpha} = \ddot{b}\ddot{c} - m\ddot{b}\ddot{\beta} \cdot m\ddot{c}\ddot{\gamma} \cdot \ddot{\alpha} = \ddot{b}\ddot{c} \cdot \left(1 - m^2\ddot{\alpha}\ddot{\beta}\ddot{\gamma}\right)$$

This leads to another symmetric result.[7]

$$\ddot{a}\ddot{d} = \ddot{b}\ddot{e} = \ddot{c}\ddot{f} \quad = \ddot{a}\ddot{b}\ddot{c} \cdot \left(1 - m^2\ddot{\alpha}\ddot{\beta}\ddot{\gamma}\right) \tag{3}$$

Indeed, hyperbolic and Euclidean hedronometry share a common characterization of perfection, namely:[8]

---

[6]The Euclidean analogue of this relation is

$$\frac{\cos\alpha}{a} = \frac{\cos\beta}{b} = \frac{\cos\gamma}{c} = \frac{1}{m}$$

[7]The Euclidean counterpart is

$$a^2 + d^2 = b^2 + e^2 = c^2 + f^2 \quad = m^2\left(\ddot{\alpha}^2 + \ddot{\beta}^2 + \ddot{\gamma}^2 - 2\ddot{\alpha}\ddot{\beta}\ddot{\gamma}\right)$$

[8]The interconnectedness of results (3) and (4) is evident in, for instance, this relation involving the Gram matrix, G:

$$\ddot{a}\ddot{d} - \ddot{b}\ddot{e} = \left(\ddot{A}\ddot{D} - \ddot{B}\ddot{E}\right)\det G$$

For maximum symmetry, we can write

$$\frac{\ddot{a}\ddot{d} - \ddot{b}\ddot{e}}{\ddot{A}\ddot{D} - \ddot{B}\ddot{E}} = \frac{\ddot{b}\ddot{e} - \ddot{c}\ddot{f}}{\ddot{B}\ddot{E} - \ddot{C}\ddot{F}} = \frac{\ddot{c}\ddot{f} - \ddot{a}\ddot{d}}{\ddot{C}\ddot{F} - \ddot{A}\ddot{D}} = \det G$$

with the understanding that, if a non-degenerate tetrahedron's measurements zero-out the numerator of a component fraction, then (as $G \neq 0$) they must also zero the corresponding denominator. The analogue of this relation in Euclidean space is

$$\frac{(a^2 + d^2) - (b^2 + e^2)}{\ddot{A}\ddot{D} - \ddot{B}\ddot{E}} = \frac{(b^2 + e^2) - (c^2 + f^2)}{\ddot{B}\ddot{E} - \ddot{C}\ddot{F}} = \frac{(c^2 + f^2) - (a^2 + d^2)}{\ddot{C}\ddot{F} - \ddot{A}\ddot{D}} = \frac{-9V^2}{8WXYZ}$$

$$\cos A \cos D = \cos B \cos E = \cos C \cos F \tag{4}$$

From the 2.5th Law of Cosines, this ultimately implies that our hyperbolic faces and pseudo-faces are related thusly (introducing a "symmetric" parameter, M):

$$\ddot{H}_2{}^2 - \ddot{H}_2\left(\ddot{W}_2\ddot{X}_2 + \ddot{Y}_2\ddot{Z}_2\right) = \ddot{J}_2{}^2 - \ddot{J}_2\left(\ddot{W}_2\ddot{Y}_2 + \ddot{Z}_2\ddot{X}_2\right) = \ddot{K}_2{}^2 - \ddot{K}_2\left(\ddot{W}_2\ddot{Z}_2 + \ddot{X}_2\ddot{Y}_2\right) =: M \tag{5}$$

In [3], the counterpart of this relation, along with the Sum of Squares identity, and the Pseudo-Heron formula for volume, gave rise to the Heron Quartic: we used the four equations to eliminate occurrences of three pseudo-face area quantities, leaving a polynomial involving only face areas and volume. As mentioned, the path to a similar result in hyperbolic space is unavailable.

Short of a volume formula, we can can provide relations that express the dihedral angle measures A, B, C, etc, in terms of $\ddot{W}_2$, $\ddot{X}_2$, $\ddot{Y}_2$, and $\ddot{Z}_2$, but each relation is a product of two 12th-degree polynomials that defy simplification efforts. Given the length of these polynomials (on the order of thousands of terms), we won't display them here, but will provide a recipe for generating them with a computer algebra system such as Mathematica.

For maximum symmetry, we set a preliminary goal to be a polynomial expressing the value M from (5) in terms of proper-face areas. From there, converting to polynomials for the pseudo-face areas, and then for dihedral angles, will be straightforward.

Now, we gather together our relevant relations: equation (5), contributing three polynomials, and the Third Law of Cosines:

```
polyH = H2^2 - H2 ( W2 X2 + Y2 Z2 ) - M
polyJ = J2^2 - J2 ( W2 Y2 + Z2 X2 ) - M
polyK = K2^2 - K2 ( W2 Z2 + X2 Y2 ) - M
loc3  = 1 - W2^2 - X2^2 - Y2^2 - Z2^2 - 4 W2 X2 Y2 Z2
          - H2^2 - J2^2 - K2^2 - 2 H2 J2 K2
          + 2 H2 ( W2 X2 + Y2 Z2 ) + 2 J2 ( W2 Y2 + Z2 X2 )
          + 2 K2 ( W2 Z2 + X2 Y2 )
```

We can make our symbol-manipulator's task a bit less labor-intensive with a few definitions to reduce symbol clutter.

$$S_X := \ddot{W}_2\ddot{X}_2 + \ddot{Y}_2\ddot{Z}_2 \qquad S_Y := \ddot{W}_2\ddot{Y}_2 + \ddot{Z}_2\ddot{X}_2 \qquad S_Z := \ddot{W}_2\ddot{Z}_2 + \ddot{X}_2\ddot{Y}_2$$

$$S := 1 - \ddot{W}_2{}^2 - \ddot{X}_2{}^2 - \ddot{Y}_2{}^2 - \ddot{Z}_2{}^2 - 4\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2$$

The transformed polynomials now feature the variables we wish to eliminate.

```
polyH = H2^2 - H2 SX - M
polyJ = J2^2 - J2 SY - M
polyK = K2^2 - K2 SZ - M
loc3  = S - H2^2 - J2^2 - K2^2 - 2 H2 J2 K2 + 2 H2 SX + 2 J2 SY + 2 K2 SZ
```

Note that we can simplify the Third Law of Cosines even further:

```
loc3  = S + H2^2 + J2^2 + K2^2 - 2 H2 J2 K2 - 6 M
```

At this point, we let our computer take over, repeatedly invoking, for instance, Mathematica's `Resultant` function to eliminate variables `H2`, `J2`, and `K2` from the four polynomials. The result(ant?) is a 12th degree polynomial in M —with over a thousand terms involving S, S_x, S_y, and S_z— that starts out innocuously enough, and has a nicely-factorable constant term, but is a nightmare in the middle (here covered over with "…"):

$$\begin{aligned} 0 &= 256M^{12} - 2304M^{11} \\ &+ 32M^{10}(243 + 48S - 68S_X^2 - 68S_Y^2 - 68S_Z^2 - 36S_XS_YS_Z) \\ &+ \ldots \\ &+ S(S+S_X^2)(S+S_Y^2)(S+S_Z^2) \\ &\quad (S+S_X^2+S_Y^2)(S+S_Y^2+S_Z^2)(S+S_Z^2+S_X^2) \\ &\quad (S+S_X^2+S_Y^2+S_Z^2-2S_XS_YS_Z) \end{aligned}$$

Expanding the S terms back into face-area terms doesn't improve things much. The number of terms in the polynomial grows to over *nineteen* thousand. However, the constant term breaks down into more factors:

$$\begin{gathered} \left(1-\ddot{W}_2{}^2-\ddot{X}_2{}^2\right)\left(1-\ddot{W}_2{}^2-\ddot{Y}_2{}^2\right)\left(1-\ddot{W}_2{}^2-\ddot{Z}_2{}^2\right) \\ \left(1-\ddot{X}_2{}^2-\ddot{Y}_2{}^2\right)\left(1-\ddot{Y}_2{}^2-\ddot{Z}_2{}^2\right)\left(1-\ddot{Z}_2{}^2-\ddot{X}_2{}^2\right) \\ \left(1-\ddot{W}_2{}^2-\ddot{X}_2{}^2-\ddot{Y}_2{}^2-\ddot{Z}_2{}^2-4\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2\right) \\ \left(1-\ddot{W}_2{}^2-\ddot{X}_2{}^2-\ddot{Y}_2{}^2-\ddot{Z}_2{}^2+(\ddot{W}_2\ddot{X}_2-\ddot{Y}_2\ddot{Z}_2)^2\right) \\ \left(1-\ddot{W}_2{}^2-\ddot{X}_2{}^2-\ddot{Y}_2{}^2-\ddot{Z}_2{}^2+(\ddot{W}_2\ddot{Y}_2-\ddot{Z}_2\ddot{X}_2)^2\right) \\ \left(1-\ddot{W}_2{}^2-\ddot{X}_2{}^2-\ddot{Y}_2{}^2-\ddot{Z}_2{}^2+(\ddot{W}_2\ddot{Z}_2-\ddot{X}_2\ddot{Y}_2)^2\right) \\ \left(\begin{array}{c} 1-\ddot{W}_2{}^2-\ddot{X}_2{}^2-\ddot{Y}_2{}^2-\ddot{Z}_2{}^2-4\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2 \\ (\ddot{W}_2\ddot{X}_2-\ddot{Y}_2\ddot{Z}_2)^2+(\ddot{W}_2\ddot{Y}_2-\ddot{Z}_2\ddot{X}_2)^2+(\ddot{W}_2\ddot{Z}_2-\ddot{X}_2\ddot{Y}_2)^2 \\ -2(\ddot{W}_2\ddot{X}_2-\ddot{Y}_2\ddot{Z}_2)(\ddot{W}_2\ddot{Y}_2-\ddot{Z}_2\ddot{X}_2)(\ddot{W}_2\ddot{Z}_2-\ddot{X}_2\ddot{Y}_2) \end{array}\right) \end{gathered}$$

Note the first few factors admit further trigonometric simplification. For instance,

$$1-\ddot{W}_2{}^2-\ddot{X}_2{}^2 = -\frac{1}{2}\left(\cos W+\cos X\right) = -\cos\left(\frac{W+X}{2}\right)\cos\left(\frac{W-X}{2}\right)$$

So far, any further sense in the M polynomial has been elusive.

Nevertheless, converting the M polynomial into, say, an $\ddot{H}_2$ polynomial is a simple as a substitution: replace M with $\ddot{H}_2{}^2 - \ddot{H}_2 S_X$. The resulting polynomial has degree 24, but factors into polynomials of degree 12 (with, respectively, 131 and 229 terms).

$$\begin{gathered} \left(16\ddot{H}_2{}^{12}-64\ddot{H}_2{}^{11}S_X+\cdots+S(S+S_Y^2)(S+S_Z^2)(S+S_Y^2+S_Z^2)\right) \\ \left(\begin{array}{c} 16\ddot{H}_2{}^{12}-128\ddot{H}_2{}^{11}S_X+\cdots \\ +(S+S_X^2)(S+S_X^2+S_Y^2)(S+S_X^2+S_Z^2)(S+S_X^2+S_Y^2+S_Z^2-2S_XS_YS_Z) \end{array}\right) \end{gathered}$$

These polynomials, though shorter than the M polynomial, aren't any less daunting to study.

But, of course, if we seek to tie back into the formula for tetrahedral volume, then we shouldn't stop at the pseudo-face level; we should proceed to the dihedral angle level. The 2.5th Law of Cosines provides the necessary bridges —for instance, $\ddot{H}_2 = \ddot{Y}_2\ddot{Z}_2 + \bar{Y}_2\bar{Z}_2\ddot{A}$— for converting our polynomial. Writing A_{XY} for $\bar{Y}_2\bar{Z}_2\ddot{A}$, we have[9]

$$\left(\begin{array}{c} 16A_{YZ}^{12} - 64A_{YZ}^{11}\left(\ddot{W}_2\ddot{X}_2 - 2\ddot{Y}_2\ddot{Z}_2\right) \\ +\cdots \\ +\bar{Y}_2{}^2\bar{Z}_2{}^2\left(1 - \ddot{W}_2{}^2 - \ddot{X}_2{}^2 - \ddot{Y}_2{}^2 + 2\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2\right) \\ \left(1 - \ddot{W}_2{}^2 - \ddot{X}_2{}^2 - \ddot{Z}_2{}^2 + 2\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2\right) \\ \left(\bar{W}_2{}^2\bar{Y}_2{}^2\bar{Z}_2{}^2 - (\ddot{X}_2 - \ddot{W}_2\ddot{Y}_2\ddot{Z}_2)^2\right) \\ \left(\bar{X}_2{}^2\bar{Y}_2{}^2\bar{Z}_2{}^2 - (\ddot{W}_2 - \ddot{X}_2\ddot{Y}_2\ddot{Z}_2)^2\right) \end{array}\right)$$

$$\left(\begin{array}{c} 16A_{YZ}^{12} + 64A_{YZ}^{11}\left(\ddot{Y}_2\ddot{Z}_2 - 2\ddot{W}_2\ddot{X}_2\right) \\ +\cdots \\ +\bar{W}_2{}^2\bar{X}_2{}^2\left(1 - \ddot{W}_2{}^2 - \ddot{Y}_2{}^2 - \ddot{Z}_2{}^2 - 2\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2\right) \\ \left(1 - \ddot{X}_2{}^2 - \ddot{Y}_2{}^2 - \ddot{Z}_2{}^2 + 2\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2\right) \\ \left(\bar{W}_2{}^2\bar{X}_2{}^2\bar{Y}_2{}^2 - (\ddot{Z}_2 - \ddot{W}_2\ddot{X}_2\ddot{Y}_2)^2\right) \\ \left(\bar{W}_2{}^2\bar{X}_2{}^2\bar{Z}_2{}^2 - (\ddot{Y}_2 - \ddot{W}_2\ddot{X}_2\ddot{Z}_2)^2\right) \end{array}\right)$$

Likewise for the other dihedral quantities, $B_{ZX} := \bar{Z}_2\bar{X}_2\ddot{B}$, $C_{XY} := \bar{X}_2\bar{Y}_2\ddot{C}$, $D_{WX} := \bar{W}_2\bar{X}_2\ddot{D}$, $E_{WY} := \bar{W}_2\bar{Y}_2\ddot{E}$, and $F_{WZ} := \bar{W}_2\bar{Z}_2\ddot{F}$. Currently, we have no information on how many roots of these polynomials may be extraneous, either inherently as individuals (being non-real values or having greater-than-one absolute values), or for failing collectively to satisfy relations such as (4). One should hope for a clear method of weeding out extraneous roots from "appropriate" ones, for otherwise, the face areas of a perfect tetrahedron provide as many as 24 "candidate" measures for each dihedral angle, determining $24^6 = 191,102,976$ different tetrahedra (or $24^6/24 = 7,962,624$, when accounting for tetrahedral symmetries). Nevertheless, we have a foundation for a strategy for computing the volume of our tetrahedron.

Theorem 4.1 (The Perfect Heron Strategy)**.** *To compute the volume of a perfect hyperbolic tetrahedron from the areas of its faces, use the family of polynomials in A_{YZ}, etc, to determine the candidates for the measures of the dihedral angles A, B, C, etc. Then substitute the "appropriate" measures into the Derevnin-Mednykh formula.*

Not the most satisfying of results.

---

[9]Despite appearances in the abbreviated display, the factors *do not* simply exchange Y and Z for W and X. The first factor, when expanded, has 672 terms; the second, 923.

On a more optimistic note: Perfection provides *definite* simplification of a tetrahedron's "hedronometric" Gram matrix representation (although pseudo-face elements remain). Multiplying each factor by one of $\ddot{H}_2$, $\ddot{J}_2$, and $\ddot{K}_2$ —and dividing out to compensate— leaves three equivalent factors.

$$\begin{aligned}
\det G \quad &:= \quad \frac{-4}{\bar{W}_2{}^2\bar{X}_2{}^2\bar{Y}_2{}^2\bar{Z}_2{}^2} \cdot \begin{array}{c} \left(\ddot{J}_2\ddot{K}_2 + \ddot{H}_2 - \ddot{W}_2\ddot{X}_2 - \ddot{Y}_2\ddot{Z}_2\right) \\ \left(\ddot{H}_2\ddot{K}_2 + \ddot{J}_2 - \ddot{W}_2\ddot{Y}_2 - \ddot{Z}_2\ddot{X}_2\right) \\ \left(\ddot{H}_2\ddot{J}_2 + \ddot{K}_2 - \ddot{W}_2\ddot{Z}_2 - \ddot{X}_2\ddot{Y}_2\right) \end{array} \\
&= \quad \frac{-4}{\bar{W}_2{}^2\bar{X}_2{}^2\bar{Y}_2{}^2\bar{Z}_2{}^2} \cdot \frac{1}{\ddot{H}_2\ddot{J}_2\ddot{K}_2} \cdot \begin{array}{c} \left(\ddot{H}_2\ddot{J}_2\ddot{K}_2 + \ddot{H}_2{}^2 - \ddot{H}_2(\ddot{W}_2\ddot{X}_2 + \ddot{Y}_2\ddot{Z}_2)\right) \\ \left(\ddot{H}_2\ddot{J}_2\ddot{K}_2 + \ddot{J}_2{}^2 - \ddot{J}_2(\ddot{W}_2\ddot{Y}_2 + \ddot{Z}_2\ddot{X}_2)\right) \\ \left(\ddot{H}_2\ddot{J}_2\ddot{K}_2 + \ddot{K}_2{}^2 - \ddot{K}_2(\ddot{W}_2\ddot{Z}_2 + \ddot{X}_2\ddot{Y}_2)\right) \end{array} \\
&= \quad \frac{-4\left(\ddot{H}_2\ddot{J}_2\ddot{K}_2 + M\right)^3}{\bar{W}_2{}^2\bar{X}_2{}^2\bar{Y}_2{}^2\bar{Z}_2{}^2\ \ddot{H}_2\ddot{J}_2\ddot{K}_2}
\end{aligned}$$

One might wish to take a further step of expressing the perfected $\det G$ in terms of face areas alone; or in terms of pseudo-face areas and M (as these values dominate the above formula). Either way may show $\det G$ to be a more suitable "symmetric" parameter than M (which was defined rather arbitrarily). This author, unfortunately, has yet to find either the patience required for his aging computer to crank through the brute-force manipulations ... or the insight required to side-step them.

5. Symmetric Tetrahedra: Using a Custom-Tailored Volume Integral

The article [2] side-steps the Derevnin-Mednykh integral (1), deriving from differential principals an integral tailored to the special case of *symmetric* hyperbolic tetrahedra, which are defined by having congruent opposing dihedral angles (say, A, B, and C); such tetrahedra have congruent faces with area W satisfying

$$\bar{W} = \frac{\ddot{A} + \ddot{B} + \ddot{C} - 1}{(1-\ddot{A})(1-\ddot{B})(1-\ddot{C})}\sqrt{1 - 2\ddot{A}\ddot{B}\ddot{C} - \ddot{A}^2 - \ddot{B}^2 - \ddot{C}^2} \tag{6}$$

The new integral is parameterized by three angles (A, B, C) meeting at a vertex (or, equivalently, surrounding a face).

$$V = \int_{v_0}^{\infty} \arcsin\frac{\ddot{A}}{\sqrt{v^2+1}} + \arcsin\frac{\ddot{B}}{\sqrt{v^2+1}} + \arcsin\frac{\ddot{C}}{\sqrt{v^2+1}} - \arcsin\frac{1}{\sqrt{v^2+1}}\ \frac{dv}{v} \tag{7}$$

where

$$v_0 = \frac{1 - 2\ddot{A}\ddot{B}\ddot{C} - \ddot{A}^2 - \ddot{B}^2 - \ddot{C}^2}{\sqrt{\left(1 - \ddot{A} + \ddot{B} + \ddot{C}\right)\left(1 + \ddot{A} - \ddot{B} + \ddot{C}\right)\left(1 + \ddot{A} + \ddot{B} - \ddot{C}\right)\left(-1 + \ddot{A} + \ddot{B} + \ddot{C}\right)}} \tag{8}$$

This result, being expressed entirely in cosines of the figure's dihedral angles, has a decidedly more hedronometry-friendly flavor than the Derevnin-Mednykh formula. We can easily use the 2.5th Law of Cosines to re-write the equation in terms of common face area (W) and pseudo-face areas (H, J, K) by making these substitutions:

$$\ddot{A} = \frac{\ddot{H}_2 - \ddot{W_2}^2}{\bar{W_2}^2} \qquad \ddot{B} = \frac{\ddot{J}_2 - \ddot{W_2}^2}{\bar{W_2}^2} \qquad \ddot{C} = \frac{\ddot{K}_2 - \ddot{W_2}^2}{\bar{W_2}^2} \tag{9}$$

However, this analysis gets more notationally compact if we work in terms of *half*-angles (A_2, B_2, C_2) and *quarter*-pseudo-faces (H_4, J_4, K_4) while shifting our trigonometric affections away from cosine and toward sine. We start by noting that (8) implies[10]

$$\bar{A}_2 = \sqrt{\frac{1 - \ddot{A}}{2}} = \sqrt{\frac{1 - \ddot{H}_2}{2\bar{W_2}^2}} = \sqrt{\frac{\bar{H_4}^2}{\bar{W_2}^2}} = \frac{\bar{H}_4}{\bar{W}_2} \qquad \bar{B}_2 = \frac{\bar{J}_4}{\bar{W}_2} \qquad \bar{C}_2 = \frac{\bar{K}_4}{\bar{W}_2} \tag{10}$$

The Third Law of Cosines provides a convenient factorization

$$0 = \left(\left(\bar{W_2}^2 - \bar{H_4}^2 - \bar{J_4}^2 - \bar{K_4}^2\right) + 2\bar{H}_4\bar{J}_4\bar{K}_4\right)\left(\left(\bar{W_2}^2 - \bar{H_4}^2 - \bar{J_4}^2 - \bar{K_4}^2\right) - 2\bar{H}_4\bar{J}_4\bar{K}_4\right)$$

that implies

$$\bar{W_2}^2 - \bar{H_4}^2 - \bar{J_4}^2 - \bar{K_4}^2 = \pm 2\bar{H}_4\bar{J}_4\bar{K}_4$$

We deduce that the "$\pm$" should be "$+$": the product $2\bar{H}_4\bar{J}_4\bar{K}_4$ is (assumed) non-negative; the left-hand side is equivalent to $\frac{1}{2}\bar{W_2}^2\left(\ddot{A} + \ddot{B} + \ddot{C} - 1\right)$, which is also non-negative.[11] Thus,

$$\bar{W_2}^2 = 2\bar{H}_4\bar{J}_4\bar{K}_4 + \bar{H_4}^2 + \bar{J_4}^2 + \bar{K_4}^2$$

so that the integration endpoint v_0 from (7) becomes

$$v_0 = \frac{\bar{H}_4\bar{J}_4\bar{K}_4\left(1 - 2\bar{H}_4\bar{J}_4\bar{K}_4 - \bar{H_4}^2 - \bar{J_4}^2 - \bar{K_4}^2\right)}{\left(2\bar{H}_4\bar{J}_4\bar{K}_4 + \bar{H_4}^2 + \bar{J_4}^2 + \bar{K_4}^2\right)\sqrt{\left(\bar{H}_4 + \bar{J}_4\bar{K}_4\right)\left(\bar{J}_4 + \bar{H}_4\bar{K}_4\right)\left(\bar{H}_4\bar{J}_4 + \bar{K}_4\right)}}$$

---

[10] Knowing that $\sin\frac{A}{2}$, $\sin\frac{B}{2}$, $\sin\frac{C}{2}$, and $\sin\frac{W}{2}$ are non-negative —because A, B, C, and W are bounded above by π— and taking $\sin\frac{H}{4}$, $\sin\frac{J}{4}$, and $\sin\frac{K}{4}$ also non-negative on the assumption that H, J, and K are bounded above by 2π.

[11] The expression $\ddot{A} + \ddot{B} + \ddot{C} - 1$ appears in (6) as a factor, among other non-negative factors, of a non-negative quantity.

We can simplify the integral itself ever-so-slightly via the change of variables $t := v\bar{W_2}^2$. With this, we have

Theorem 5.1 (The Pseudo-Heron Formula for Volume of a Symmetric Hyperbolic Tetrahedron)**.** *A symmetric hyperbolic tetrahedron with pseudo-face areas H, J, and K (and face area W) has volume given by*

(11)
$$V = \int_{t_0}^{\infty} \arcsin\frac{\bar{W_2}^2 - 2\bar{H_4}^2}{\sqrt{t^2+\bar{W_2}^4}} + \arcsin\frac{\bar{W_2}^2 - 2\bar{J_4}^2}{\sqrt{t^2+\bar{W_2}^4}} + \arcsin\frac{\bar{W_2}^2 - 2\bar{K_4}^2}{\sqrt{t^2+\bar{W_2}^4}} - \arcsin\frac{\bar{W_2}^2}{\sqrt{t^2+\bar{W_2}^4}}\;\frac{dt}{t}$$

where

$$\bar{H_4} := \sin\frac{H}{4} \qquad \bar{J_4} := \sin\frac{J}{4} \qquad \bar{K_4} := \sin\frac{K}{4} \qquad \bar{W_2}^2 := \sin^2\frac{W}{2} = 2\bar{H_4}\bar{J_4}\bar{K_4} + \bar{H_4}^2 + \bar{J_4}^2 + \bar{K_4}^2$$
$$t_0 := \frac{\bar{H_4}\bar{J_4}\bar{K_4}\left(1 - 2\bar{H_4}\bar{J_4}\bar{K_4} - \bar{H_4}^2 - \bar{J_4}^2 - \bar{K_4}^2\right)}{\sqrt{\left(\bar{H_4}+\bar{J_4}\bar{K_4}\right)\left(\bar{J_4}+\bar{K_4}\bar{H_4}\right)\left(\bar{K_4}+\bar{H_4}\bar{J_4}\right)}}$$

Given that $\bar{W_2}$ can be expressed in terms of H_4, J_4, and K_4, the formula is effectively "pure pseudo". In Euclidean space, the corresponding formulas for an "equihedral" tetrahedron are $W = \frac{1}{2}\sqrt{H^2+J^2+K^2}$ and $V = \frac{1}{3}\sqrt{HJK}$. (The simplicity of the Euclidean volume result raises the question: Does the hyperbolic volume integral have a simpler representation?)

Let us see how perfection affects this scenario.

A perfect symmetric tetrahedron necessarily has six congruent dihedral angles.[12] Consequently, its three pseudo-face areas are equal.[13] This gives a reduced volume formula of the form

$$V = \int_{t_0}^{\infty} 3\arcsin\frac{\bar{W_2}^2 - 2\bar{H_4}^2}{\sqrt{t^2+\bar{W_2}^4}} - \arcsin\frac{\bar{W_2}^2}{\sqrt{t^2+\bar{W_2}^4}}\;\frac{dt}{t} \tag{12}$$

where

$$\bar{W_2}^2 = 2\bar{H_4}^3 + 3\bar{H_4}^2$$
$$t_0 := \frac{\bar{H_4}^2\left(1+\bar{H_4}\right)\left(1-2\bar{H_4}\right)}{\sqrt{\bar{H_4}\left(1+\bar{H_4}\right)}} = \bar{H_4}\left(1-2\bar{H_4}\right)\sqrt{\bar{H_4}\left(1+\bar{H_4}\right)}$$

The reduced Third Law of Cosines reduces further to provide a simpler bridge between the face area and pseudo-face area.

$$\bar{W_2}^2 = 2\bar{H_4}^3 + 3\bar{H_4}^2$$

We can thus solve for H in terms of W, generating three candidates:

---

[12]Equality of opposite dihedral angles, coupled with the perfection condition (4), guarantees this.

[13]Without a geometric interpretation of pseudo-faces, the word "congruent" isn't appropriate.

$$\bar{H}_4 = \frac{1}{2}\left(2\cos\frac{W-\pi}{3} - 1\right), \quad \frac{1}{2}\left(2\cos\frac{W+\pi}{3} - 1\right), \quad -\frac{1}{2}\left(1 + 2\cos\frac{W}{3}\right)$$

The area of the hyperbolic triangle W is bounded above by π, and the area of the pseudo-face H is bounded above by 2π so that $\bar{H}_4$ must be non-negative. This eliminates all candidate values except the first. From this we deduce the following:

Theorem 5.2 (The Heron Formula for Volume of a Perfect Symmetric Hyperbolic Tetrahedron)**.** *A perfect symmetric hyperbolic tetrahedron with face area W (and pseudo-face area H) has volume given by*

$$V = \int_{t_0}^{\infty} 3\arcsin\frac{\bar{W_2}^2 - 2\bar{H_4}^2}{\sqrt{t^2 + \bar{W_2}^2}} - \arcsin\frac{\bar{W_2}^2}{\sqrt{t^2 + \bar{W_2}^2}} \;\; \frac{dt}{t} \tag{13}$$

where

$$\bar{W}_2 := \sin\frac{W}{2} \qquad \bar{H}_4 := \sin\frac{H}{4} = \frac{1}{2}\left(2\cos\frac{W-\pi}{3} - 1\right)$$

and

$$t_0 = 2\left(2\cos\frac{W-\pi}{3} - 1\right)\sin^2\frac{W-\pi}{6}\sqrt{\sin\frac{W}{3}\sin\frac{W+\pi}{3}}$$

Again, the simplicity of the Euclidean counterpart formulas —$H^2 = \frac{4}{3}W^2$ and $V^4 = \frac{64}{2187}W^6$— suggests that this result has a more elegant presentation.

References

[1] D.A. Derevnin and A.D. Mednykh. "A formula for the volume of a hyperbolic tetrahedon". Communications of the Moscow Mathematical Society 60, No.2 (2005), pp. 346-348.
`http://mathlab.snu.ac.kr/~top/articles/MednykhDerevninHyperbolicTetrahedron.pdf`
(See also the Hyperbolic Volume Workshop Homepage: `http://mathlab.snu.ac.kr/~top/`)

[2] D.A. Derevnin, A.D. Mednykh, M.G. Pashkevich. "On the volume of symmetric tetrahedron". Siberian Mathematical Journal, Vol. 45, No. 5 (2004), pp. 840-848.
`http://mathlab.snu.ac.kr/~top/articles/Tetr_Volume.pdf`

[3] B.D.S. McConnell. "Heron-Like Results for Tetrahedral Volume".
`http://www.daylateanddollarshort.com/math/pdfs/heron4tet.pdf` (2008).

[4] B.D.S. McConnell. "The Laws of Cosines for Non-Euclidean Tetrahedra".
`http://www.daylateanddollarshort.com/math/pdfs/loc4net.pdf` (2006)

[5] B.D.S. McConnell. "Pseudofaces of Tetrahedra".
`http://www.daylateanddollarshort.com/math/pdfs/pseudofaces.pdf` (2006)

B.D.S. "Don" McConnell
`math@daylateanddollarshort.com`