PSEUDOFACES OF TETRAHEDRA

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The Law of Cosines for tetrahedra states that

$$W^2 = X^2 + Y^2 + Z^2 - 2YZ\cos\angle OA - 2ZX\cos\angle OB - 2XY\cos\angle OC$$

where $W := area\triangle ABC$, $X := area\triangle OBC$, $Y := area\triangle OCA$, and $Z := area\triangle OAB$, and where $\angle OA$, $\angle OB$, and $\angle OC$ are the dihedral angles along corresponding edges. The resemblance to the Law of Cosines for triangle is already striking, but can be made moreso.

Combine the above equation with its counterparts

$$\begin{aligned}
X^2 &= W^2 + Y^2 + Z^2 - 2YZ\cos\angle OA - 2ZW\cos\angle AB - 2WY\cos\angle CA \\
Y^2 &= X^2 + W^2 + Z^2 - 2WZ\cos\angle AB - 2ZX\cos\angle OB - 2XW\cos\angle BC \\
Z^2 &= X^2 + Y^2 + W^2 - 2YW\cos\angle CA - 2WX\cos\angle BC - 2XY\cos\angle OC
\end{aligned}$$

to get a family of relations

$$\begin{aligned}
Y^2 + Z^2 - 2YZ\cos\angle OA &= W^2 + X^2 - 2WX\cos\angle BC \\
Z^2 + X^2 - 2ZX\cos\angle OB &= W^2 + Y^2 - 2WY\cos\angle CA \\
X^2 + Y^2 - 2XY\cos\angle OC &= W^2 + Z^2 - 2WZ\cos\angle AB
\end{aligned}$$

that teeter on the very brink of exactly matching the triangular Law of Cosines. We push them over that brink by formally defining P, Q, and R such that

$$\begin{aligned}
Y^2 + Z^2 - 2YZ\cos\angle OA &= P^2 = W^2 + X^2 - 2WX\cos\angle BC \\
Z^2 + X^2 - 2ZX\cos\angle OB &= Q^2 = W^2 + Y^2 - 2WY\cos\angle CA \\
X^2 + Y^2 - 2XY\cos\angle OC &= R^2 = W^2 + Z^2 - 2WZ\cos\angle AB
\end{aligned}$$

We say that P, Q, and R are the areas of "pseudofaces" of the tetrahedron ... whatever "pseudofaces" might be. Note that, although pseudo, these faces together are as good as the "real" faces (at least when squared):

$$P^2 + Q^2 + R^2 = W^2 + X^2 + Y^2 + Z^2$$

With judicious substitution and rearrangement, one can use pseudofaces to convert a "hedronometric" volume formula that favors one vertex

$$V^4 = \frac{4}{81}X^2Y^2Z^2\begin{pmatrix} 1 - 2\cos(\angle OA)\cos(\angle OB)\cos(\angle OC) \\ -\cos^2(\angle OA) - \cos^2(\angle OB) - \cos^2(\angle OC) \end{pmatrix}$$

Date: 6 March, 2005.

into something more symmetrical

$$V^4 = \frac{1}{81}\left(\begin{array}{c} 2W^2X^2Y^2 + 2W^2X^2Z^2 + 2W^2Y^2X^2 + 2X^2Y^2Z^2 + P^2Q^2R^2 \\ -P^2\left(W^2X^2 + Y^2Z^2\right) - Q^2\left(W^2Y^2 + Z^2X^2\right) - R^2\left(W^2Z^2 + X^2Y^2\right) \end{array}\right)$$

So, here we see that the collection of faces and pseudofaces determine a tetrahedron (and its volume) uniquely. This type of result isn't true for just faces alone.

Pseudofaces revealed

Although initially defined merely to make a few equations look nice, pseudofaces in fact have a geometric interpretation: viewing a tetrahedron in a direction perpendicular to the plane determined by opposite edges (for instance OA and BC, as in Figure (1a) below) reveals a quadrilateral. That quadrilateral is a pseudoface, insofar as its area is computed via a pseudoface formula.

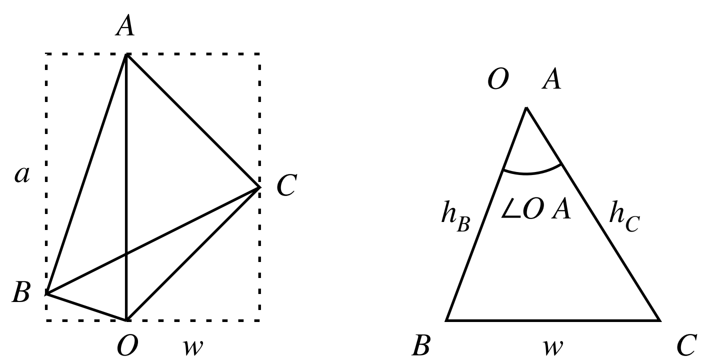

Figure 1. Views of a Tetrahedron. (a) Perpendicular to OA and BC; (b) parallel to OA.

Figure (1a) surrounds a pseudoface with a rectangle of dimensions $a := |OA|$ and w, so that the area of the pseudoface satisfies

$$P^2 = \frac{1}{4}a^2w^2$$

From Figure (1b), we compute the distance w from the triangular Law of Cosines, using the dihedral angle $\angle OA$ and the lengths h_B and h_C (which are altitudes to OA of $\triangle OAB$ and $\triangle OAC$, respectively):

$$w^2 = h_B^2 + h_C^2 - 2h_Bh_Ccos\angle OA$$

Thus,

$$\begin{aligned} P^2 &= \frac{1}{4}a^2\left(h_B^2 + h_C^2 - 2h_B h_C \cos\angle OA\right) \\ &= \frac{1}{4}a^2 h_B^2 + \frac{1}{4}a^2 h_C^2 - 2\cdot\frac{1}{2}a h_B \cdot \frac{1}{2}a h_C \cdot \cos\angle OA \\ &= Y^2 + Z^2 - 2YZ\cos\angle OA \end{aligned}$$

A slightly different analysis, involving sighting the tetrahedron along BC, would yield

$$P^2 = W^2 + X^2 - 2WX\cos\angle BC$$

Projecting the tetrahedron into planes determined by the other two pairs of opposite edges gives pseudofaces Q and R.

Assorted Pseudoface Formulas

A Volume Formula. If p is the distance between edges OA and BC —the "pseudoaltitude" corresponding to the pseudoface P— then

$$V = \frac{1}{3}pP$$

Proof. Project points B and C to points B' and C' in the plane through OA parallel to P, and project points O and A to points O' and A' in the plane through BC parallel to P. Then $OB'AC'$ and $O'BA'C$ are copies of the pseudoface P, and form bases of a right prism with height p and volume pP. We can carve the tetrahedron out of the prism by removing the height-p tetrahedron pairs $OO'BC$ and $AA'BC$ (whose bases together form $O'BA'C$), and $BB'OA$ and $CC'OA$ (whose bases together form $OB'AC'$). Each of these tetrahedron pairs has one-third the volume of the prism, so that the original tetrahedron makes up the last third of that volume. □

Cosine Formulas. We naturally define "the" angle between two (real) faces of a tetrahedron as the dihedral angle containing the interior of the tetrahedron (or as the supplement of the angle between two inward-pointing —or two outward-pointing— normal vectors). Pseudofaces have no inherent position relative to the interior of the tetrahedron, and so we cannot define the angle between pseudofaces without ambiguity or artifice. We will accept the ambiguity and let vectors provide the artifice.

Recall that, for vectors $\mathbf{u}$ and $\mathbf{v}$ subtending angle θ,

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \qquad \text{and} \qquad \|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$$

and that $\mathbf{u}\times\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Thus, we have formulas such as these:

$$X^2 = \frac{1}{4}\|\mathbf{b}\times\mathbf{c}\|^2 \qquad Y^2 = \frac{1}{4}\|\mathbf{c}\times\mathbf{a}\|^2 \qquad Z^2 = \frac{1}{4}\|\mathbf{a}\times\mathbf{b}\|^2 \qquad W^2 = \frac{1}{4}\|(\mathbf{a}-\mathbf{b})\times(\mathbf{c}-\mathbf{b})\|^2$$

$$YZ\cos\angle OA = -\frac{1}{4}\left(\mathbf{c}\times\mathbf{a}\right)\cdot\left(\mathbf{a}\times\mathbf{b}\right) \qquad ZX\cos\angle OB = -\frac{1}{4}\left(\mathbf{a}\times\mathbf{b}\right)\cdot\left(\mathbf{b}\times\mathbf{c}\right)$$

$$XY\cos\angle OC = -\frac{1}{4}\left(\mathbf{b}\times\mathbf{c}\right)\cdot\left(\mathbf{c}\times\mathbf{a}\right)$$

where $\mathbf{a} := \overrightarrow{OA}$, $\mathbf{b} := \overrightarrow{OB}$, and $\mathbf{c} := \overrightarrow{OC}$.

(Note that, although there are more direct approaches, one can recover the Law of Cosines for Tetrahedra by expanding the vector formula for W^2 in terms of the other expressions.)

With pseudofaces, we have these unambiguous area formulas

$$P^2 = \frac{1}{4}\|\mathbf{a} \times (\mathbf{c} - \mathbf{b})\|^2 \qquad Q^2 = \frac{1}{4}\|\mathbf{b} \times (\mathbf{a} - \mathbf{c})\|^2 \qquad R^2 = \frac{1}{4}\|\mathbf{c} \times (\mathbf{b} - \mathbf{a})\|^2$$

For angles, we make a subjective selection of normal vector (which depends upon our preference for vertex O), and declare the angle between two pseudofaces is the angle (not the supplement of the angle) between the two corresponding normal vectors. In particular, writing θ_P for the preferred angle between pseudofaces Q and R, we have

$$\begin{aligned} QR\cos\theta_P &:= -\frac{1}{4}\left(\mathbf{b} \times (\mathbf{a} - \mathbf{c})\right) \cdot \left(\mathbf{c} \times (\mathbf{b} - \mathbf{a})\right) \\ &= \ldots \\ &= -\frac{1}{4}\left((\mathbf{c} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{a}) \cdot (\mathbf{a} \times \mathbf{c})\right. \\ &\qquad \left. + (\mathbf{c} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{c} \times \mathbf{b})\right) \\ &= -X^2 + YZ\cos\angle OA + ZX\cos\angle OB + XY\cos\angle OC \end{aligned}$$

so that

$$\begin{aligned} 2QR\cos\theta_P &= -2X^2 + (2YZ\cos\angle OA + 2ZX\cos\angle OB + 2XY\cos\angle OC) \\ &= -2X^2 + \left(X^2 + Y^2 + Z^2 - W^2\right) \\ &= \left(Y^2 + Z^2\right) - \left(W^2 + X^2\right) \\ &= \left(P^2 + 2YZ\cos OA\right) - \left(P^2 + 2WX\cos BC\right) \\ &= 2\left(YZ\cos OA - WX\cos BC\right) \end{aligned}$$

Altogether, we have

$$\begin{aligned} \left(Y^2 + Z^2\right) - \left(W^2 + X^2\right) &= 2QR\cos\theta_P = 2\left(YZ\cos OA - WX\cos BC\right) \\ \left(Z^2 + X^2\right) - \left(W^2 + Y^2\right) &= 2RP\cos\theta_Q = 2\left(ZX\cos OB - WY\cos CA\right) \\ \left(X^2 + Y^2\right) - \left(W^2 + Z^2\right) &= 2PQ\cos\theta_R = 2\left(XY\cos OC - WZ\cos AB\right) \end{aligned}$$

With other choices in how we define the angles θ_P, θ_Q, and θ_R, the above formulas could be off by a sign (replacing an angle by its supplement changes the sign of the cosine); the formulas, therefore, are only "unambiguously correct" in absolute value. Note that we do have this unambiguous corollary: *The pseudofaces of an equihedral tetrahedron — $W = X = Y = Z$ — are mutually perpendicular.*

Finally, what good are cosines without a corresponding Law of Cosines?

$$\begin{aligned}
Q^2+R^2-2QR\cos\theta_P &= \big(Z^2+X^2-2ZX\cos OB\big)+\big(X^2+Y^2-XY\cos OC\big)\\
&\qquad +\big(Y^2+Z^2\big)-\big(W^2+X^2\big)\\
&= Y^2+Z^2-W^2\\
&\qquad +\big(X^2+Y^2+Z^2-2ZX\cos OB-2XY\cos OC\big)\\
&= Y^2+Z^2-W^2+\big(W^2+2YZ\cos OA\big)\\
&= Y^2+Z^2+2YZ\cos OA\\
R^2+P^2-2RP\cos\theta_Q &= Z^2+X^2+2ZX\cos OB\\
P^2+Q^2-2PQ\cos\theta_R &= X^2+Y^2+2XY\cos OC
\end{aligned}$$

Note that

$$\begin{aligned}
Q^2+R^2-2QR\cos(\text{supplement of }\theta_P) &= W^2+X^2+2WX\cos BC\\
R^2+P^2-2RP\cos(\text{supplement of }\theta_Q) &= W^2+Y^2+2WY\cos CA\\
P^2+Q^2-2PQ\cos(\text{supplement of }\theta_R) &= W^2+Z^2+2WZ\cos AB
\end{aligned}$$

We leave the reader to investigate "second-pseudofaces" whose areas satisfy, say,

$$H^2 := Q^2+R^2-2QR\cos\theta_P$$

although we will point out the geometric interpretation of such elements. They are pseudofaces of "exterior" tetrahedra to the given tetrahedron: Reflect point C in segment OA to get C' and a new tetrahedron $OABC'$ that shares face OAB with the original tetrahedron. The faces OAC' (which is congruent to OAC) and OAB enclose the supplement of angle bounded by OAC and OAB, hence the pseudoface corresponding to OA and BC' has area given by

$$Y^2+Z^2-2YZ\cos(\text{supp. }\angle OA)=Y^2+Z^2+2YZ\cos OA=H^2$$

Whether these exterior tetrahedra attach to edges surrounding a vertex (O) or to edges surrounding a face ($\triangle ABC$) depends upon the preferred definition of the pseudoface angles.

Another Volume Formula. Making appropriate substitutions into the hedronometric volume formula from the first section of this article, we can achieve this volume formula

$$V^4=\frac{1}{81}P^2Q^2R^2\left(\begin{array}{c}1+2\cos\theta_P\cos\theta_Q\cos\theta_R\\-\cos^2\theta_P-\cos^2\theta_Q-\cos^2\theta_R\end{array}\right)$$

(The $2\cos\theta_P\cos\theta_Q\cos\theta_R$ term is subject to sign change under alternate definition of the angles involved.)

An equihedral tetrahedron's volume, therefore, (unambiguously) satisfies

$$V^2=\frac{1}{9}PQR$$

For example, a regular tetrahedron with edge-length s has pseudofaces in the form of squares of edge length $s\sqrt{2}/2$ and area $s^2/2$. Thus, the tetrahedron's volume is $\sqrt{s^6/72} = s^3\sqrt{2}/12$.

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