SPECTRAL REALIZATIONS OF GRAPHS

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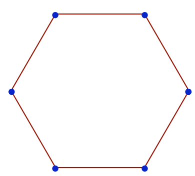
1. Introduction

The boundary of the regular hexagon in $\mathbb{R}^2$ and the vertex-and-edge skeleton of the regular tetrahedron in $\mathbb{R}^3$, as geometric realizations of the combinatorial 6-cycle and complete graph on 4 vertices, exhibit a significant property: Each automorphism of the graph induces a "rigid" isometry of the figure. We call such a figure *harmonious*.[1]



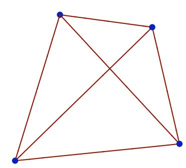


Figure 1. A pair of harmonious graph realizations (assuming the latter in 3D).

Harmonious realizations can have considerable value as aids to the intuitive understanding of the graph's structure, but such realizations are generally elusive. This note explains and explores a proposition that provides a straightforward way to generate an entire family of harmonious realizations of any graph:

> *A matrix whose* rows *form an orthogonal basis of an eigenspace of a graph's adjacency matrix has* columns *that serve as coordinate vectors of the vertices of an harmonious realization of the graph. This is a (projection of a)* **spectral realization**.

The hundreds of diagrams in Section 4[2] illustrate that spectral realizations of graphs with a high degree of symmetry can have great visual appeal. Or, not: they may exist in arbitrarily-high-dimensional spaces, or they may appear as an uninspiring jumble of points in one-dimensional space. In most cases, they collapse vertices and even edges into single points, and are therefore only very rarely *faithful*. Nevertheless, spectral realizations can often provide useful starting points for visualization efforts. (A basic *Mathematica* recipe for computing (projected) spectral realizations appears at the end of Section 3.)

Not every harmonious realization of a graph is spectral. For instance, many Archimedean polyhedra represent harmonious but non-spectral realizations of their skeleta. However, spectral realizations

[1]This terminology follows [1], and is not to be confused with "harmonious graph" [3].

[2]See also contributions (to appear) in the Wolfram Demonstrations Project (http://demonstrations.wolfram.com). As of this writing, the author has prepared "Spectral Realizations of Polyhedral Skeleta" for submission, and is working on a more general "Spectral Realizations of Graphs". These interactive, *Mathematica*-based notebooks are also available directly from the author.

have added significance (to be taken up in more detail in another paper) as building blocks of *all possible* realizations.

2. Definitions and Notation

For the purposes of this paper, a (combinatorial) graph, G, consists of a finite set of vertices and a set of edges *uniquely* defined by adjacency relations between distinct pairs of vertices. (By this definition, each pair of vertices determine at most one edge, and each edge determines exactly two endpoint vertices.) A (geometric) *realization* of G —denoted $\widehat{G}$— represents the vertices of G as (not-necessarily-distinct) points in some $\mathbb{R}^m$; this induces a representation of edges of G as (not-necessarily-distinct, and very-likely-degenerate) line segments joining pairs of those points. We formally ignore any two-, three-, or higher-dimensional regions that the edges may seem to surround.

Our results regarding harmonious and spectral graph realizations amount to gleanings from matrix algebra. We therefore cast our objects of concern into an appropriate context.

2.1. **Matrixification.**

Fixing an order on the n vertices of G, we encode the graph's structure in the (symmetric) n-by-n *adjacency matrix*, $[G]$, whose (i,j)-th entry counts the number (0 or 1) of edges joining the i-th vertex to the j-th vertex. We encode the realization $\widehat{G}$ in the m-by-n *coordinate matrix*, $[\widehat{G}]$, whose i-th column is the coordinate vector of G's i-th vertex in $\mathbb{R}^m$.

A (vertex) permutation, P, gives rise to an n-by-n *permutation matrix*, $[P]$, such that multiplication of a matrix *on the right* by $[P]$ permutes a matrix's columns in the manner prescribed by P. (Multiplying *on the left* by $[P]^\top$ permutes the rows in the same manner.)

Not all vertex permutations are graph automorphisms. A graph automorphism must preserve vertex adjacency; equivalently, permuting the columns *and rows* of the adjacency matrix in accordance with an automorphism should preserve that matrix. Thus,

$$P \text{ is an automorphism of } G \quad \Leftrightarrow \quad [P]^\top[G][P] = [G]$$

Permutation matrices being orthogonal,[3] the above asserts the commutativity of $[P]$ and $[G]$:

$$P \text{ is an automorphism of } G \quad \Leftrightarrow \quad [G][P] = [P][G]$$

We matrixify a linear transformation of $\mathbb{R}^m$ in the traditional way, as an m-by-m matrix whose i-th column is the image of the i-th column of $\mathbf{I}$. Multiplying the coordinate matrix of a realization *on the left* by a transformation matrix yields the coordinate matrix of the transformed realization. The transformation is "rigid" if (and only if) its matrix is orthogonal.

Finally, to apply a translation to all n points of a realization, we add to the coordinate matrix a *translation matrix* of the form $\mathbf{t}\mathbf{1}_n^\top$, where $\mathbf{t}$ is the m-coordinate column vector of the translation, and where $\mathbf{1}_n$ is the "all 1s" column vector of length n.

[3] $[P]^\top[P] = [P][P]^\top = \mathbf{I}$

2.2. **Harmonious Realizations.** We can now define a harmonious realization in terms of matrix relations:

Definition 1. *A realization, $\widehat{G}$, of a graph, G, is harmonious if and only if, for each vertex permutation P,*

$$\left(\ [G][P] = [P][G]\ \right) \implies \left(\ [\widehat{G}][P] = \mathbf{Q}[\widehat{G}] + \mathbf{t}\mathbf{1}_n^\top\ \right) \tag{1}$$

for some orthogonal matrix $\mathbf{Q}$ and some vector $\mathbf{t}$, both of which depend upon P.

Here, the left-hand side of the implication supposes P is an automorphism of G. The right-hand side indicates that permuting the vertices of the realization $\widehat{G}$ can be achieved by applying a rigid motion —a linear isometry and a translation— to the realization.[4] This note's featured realizations have no translational component, which allows us focus on a slightly simpler sufficient condition for harmony:

Corollary 1. *A graph realization is harmonious if, for each vertex permutation P,*

$$\left(\ [G][P] = [P][G]\ \right) \implies \left(\ [\widehat{G}][P] = \mathbf{Q}[\widehat{G}]\ \right) \tag{3}$$

where $\mathbf{Q}$ is orthogonal and depends upon P.

3. Harmonious Realizations

The condition (3) for realization harmony leads immediately to some key observations:

Lemma 1. *The following are matrices of harmonious realizations of the n-vertex graph G in $\mathbb{R}^n$.*

(a) *The n-by-n identity matrix, $\mathbf{I}$.*
(b) *The adjacency matrix $[G]$.*
(c) *$p([G])$ for any polynomial p with real coefficients.*

Proof. In (3), take $\mathbf{Q} := [P]$; that is, simply take the isometry to be the one that permutes the coordinate axes appropriately. Then (3), effectively, becomes the statement "if $[P]$ commutes with $[G]$, then $[P]$ commutes with $[\widehat{G}]$". This implication is trivially true for the $[\widehat{G}]$ candidates proposed in (a) and (b), and it is only slightly non-trivially true for the candidate in (c). □

Spectral realizations are special cases of part (c).

[4]In point of fact, the translation vector $\mathbf{t}$ in the definition depends upon $[\widehat{G}]$ and $\mathbf{Q}$. The realizations $[\widehat{G}]$ and $[\widehat{G}][P]$ share vertices, hence also share a *center* that we define and compute as $\frac{1}{n}[\widehat{G}]\mathbf{1}_n$, the average of the coordinate vectors of the vertices. The realization $\mathbf{Q}[\widehat{G}] + \mathbf{t}\mathbf{1}_n^\top$ has center $\frac{1}{n}\left(\mathbf{Q}[\widehat{G}] + \mathbf{t}\mathbf{1}_n^\top\right)\mathbf{1}_n = \frac{1}{n}\mathbf{Q}[\widehat{G}]\mathbf{1}_n + \mathbf{t}$. This must coincide with the original center, and so

$$\mathbf{t} = \frac{1}{n}[\widehat{G}]\mathbf{1}_n - \frac{1}{n}\mathbf{Q}[\widehat{G}]\mathbf{1}_n = \frac{1}{n}\left(\mathbf{I} - \mathbf{Q}\right)[\widehat{G}]\mathbf{1}_n \tag{2}$$

Note that, as a consequence, any realization that requires a translation component to geometrically "realize" an automorphism is the translate of a realization that requires no such translation. (Translation-free realizations need not be centered at the origin, however.) Thus, if we are interested in characterizing the *shape* of an harmonious realization, we could ignore the translation component in Definition 1.

3.1. **The Spectral Realizations.** Basic results from linear algebra —namely, the Spectral Theorem for Real Symmetric Matrices, and its consequences— provide these facts about the adjacency matrix of an n-vertex graph G:

- $[G]$'s *spectrum* comprises real eigenvalues, which we will denote $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_k$.
- $\mathbb{R}^n$ admits an orthonormal basis of eigenvectors of $[G]$. We collect such a basis into the columns of the (orthogonal) n-by-n matrix $\mathbf{B}$.
- Consequently, $[G]$ is diagonalizable by $\mathbf{B}$: $[G] = \mathbf{B}\mathbf{D}\mathbf{B}^\top$, where $\mathbf{D}$ is a diagonal matrix whose diagonal entries are the λ_i.

With the first of these facts in mind, we make an unmotivated definition.

Definition 2. *The* spectral λ_i-realization of G *has coordinate matrix,* $\mathbf{\Lambda}_i$*, given by*

$$\mathbf{\Lambda}_i := p_i([G]), \qquad \text{where} \qquad p_i(x) := \prod_{j\neq i} \frac{(x-\lambda_j)}{(\lambda_i - \lambda_j)} \tag{4}$$

As polynomials in $[G]$, spectral realizations are necessarily harmonious (specifically such that $\mathbf{Q} := [P]$ in (3)). Their relation to the eigen-properties of $[G]$ make their coordinate matrices straightforward to generate and analyze. Before confronting the notion that spectral realizations, encoded by n-by-n coordinate matrices, naturally reside in $\mathbb{R}^n$, we note in passing two consequences of the fact that $\mathbf{\Lambda}_i$ is a λ_i-eigenmatrix of $[G]$:

- The result of replacing each vertex with the (vector) sum of its neighbors is equivalent to the result of scaling the realization by a factor of λ_i relative to the origin. (This is nothing more than an interpretation of the equation $\mathbf{\Lambda}_i[G] = \lambda_i \mathbf{\Lambda}_i$, where $[G]$ is seen as acting on $\mathbf{\Lambda}_i$ to compute the requisite vector sums, and, of course, the λ_i acts to scale the vectors as described.) We call a realization with this property *eigenic*.

- The relation $\mathbf{I} = \sum \mathbf{\Lambda}_i$ (the unique decomposition of $\mathbf{I}$ as the sum of eigenmatrices of $[G]$) implies that $\mathbf{M} = \sum \mathbf{M}\mathbf{\Lambda}_i$ for any n-column matrix $\mathbf{M}$. Interpreting $\mathbf{M}$ on the left-hand side as a coordinate matrix, and on the right-hand side as transformation matrix applied to $\mathbf{\Lambda}_i$, we have that *any realization is the* vertex sum[5] *of uniquely-determined linear images of spectral realizations.*

  We can (and, in a separate paper, *will*) show that any matrix is the sum of (not-necessarily-uniquely-determined) scaled-orthogonal matrices, so that, indeed, *any realization is the vertex sum of* similar *images of the spectral realizations.*

With that digression completed, we conclude our exposition by confirming the validity of our proposed means of projecting spectral realizations into smaller-dimensional space.

3.2. **Spectral Realizations in Fewer Dimensions.** The diagonalizability of $[G]$ implies that, with $\mathbf{B}$ and p_i as given in the preceding subsection, $\mathbf{\Lambda}_i = \mathbf{B}p_i(\mathbf{D})\mathbf{B}^\top$. The matrix $p_i(\mathbf{D})$ is diagonal, with entries $p_i(\lambda_j)$ that are either 1 (when $i = j$) or 0 (when $i \neq j$), so that, in fact,

$$\mathbf{\Lambda}_i = \mathbf{B}_i\mathbf{B}_i^\top \tag{5}$$

[5] Vertex-wise vector sum, induced by the sum of coordinate matrices.

where $\mathbf{B}_i$ consists of the λ_i-eigenvector columns from $\mathbf{B}$. (Note that this equality is independent of the choice of basis vectors that make up $\mathbf{B}_i$ (or $\mathbf{B}$).) This factorization leads to the result referenced in the introduction of this paper:

Theorem 2. *Given a matrix $\mathbf{B}_i$ whose columns constitute an orthogonal basis of the λ_i-eigenspace of $[G]$, the realization of G with coordinate matrix $\mathbf{B}_i^\top$ is an isometric image of the spectral λ_i-realization of G embedded into $\mathbb{R}^r$, where r is the rank of $\mathbf{B}_i$.*

Proof. The significance of the rank r is clear. To show the isometric relation to the spectral realization, we need only exploit the orthogonality of the columns of $\mathbf{B}_i$ —via $\mathbf{B}_i^\top \mathbf{B}_i = \mathbf{I}$— to show equality of products of the form $\mathbf{M}^\top\mathbf{M}$ for matrices $\mathbf{B}_i^\top$ and $\mathbf{\Lambda}_i$:

$$\mathbf{\Lambda}_i \rightarrow \quad \mathbf{\Lambda}_i^\top \mathbf{\Lambda}_i = \left(\mathbf{B}_i\mathbf{B}_i^\top\right)^\top \left(\mathbf{B}_i\mathbf{B}_i^\top\right) = \mathbf{B}_i\mathbf{B}_i^\top\mathbf{B}_i\mathbf{B}_i^\top = \mathbf{B}_i\mathbf{B}_i^\top = \left(\mathbf{B}_i^\top\right)^\top \mathbf{B}_i^\top \quad \leftarrow \mathbf{B}_i^\top \tag{6}$$

$\square$

Figure 2 leverages this Theorem —and *Mathematica*— to automate the search for spectral realizations of a graph: compute the eigenvectors of the adjacency matrix, orthogonalize the lot of them, and collect them according to eigenvalue.

```
(* Define 'adjMat' as an adjacency matrix. (Here, it's the skeleton of the cube.) *)
adjMat = GraphData["CubicalGraph", "AdjacencyMatrix"];

(* Get eigenvalues and eigenvectors. (Be prepared to wait!) *)
{valList, vecList} = Eigensystem[adjMat];
(* Sort values by eigendimension, and remove duplicates *)
vals = Union[Sort[valList, Count[valList, #1] < Count[valList, #2] &]];
(* Orthogonalize all eigenspaces at once. (Another lengthy process!) *)
vecList = Orthogonalize[vecList];

(* Build eigenmatrices from rows corresponding to each eigenvalue *)
matList = Map[vecList[[Flatten[Position[valList, #1]]]] &, vals];

(* Display results: eigenvalue, dimension, coordinate matrix, and plot *)
Table[
 Column[{
   vals[[k]],
   Count[valList, vals[[k]]],
   MatrixForm[matList[[k]]],
   If[Count[valList, vals[[k]]] < 3,
    GraphPlot[adjMat,
     VertexCoordinateRules ->
      If[Count[valList, vals[[k]]] < 2,
       Map[PadRight[#1, 2] &, Transpose[matList[[k]]]],
       Transpose[matList[[k]]]]],
    GraphPlot3D[adjMat,
     VertexCoordinateRules ->
      Transpose[matList[[k]][[1 ;; 3]]]]]}], {k, Length[vals]}]
```

FIGURE 2. Computationally expensive *Mathematica* recipe for generating coordinate matrices (and displaying 2- or 3-dimensional plots) of all (projected) spectral realizations of a graph. (Strategic use of `N[]` recommended.)

Note that the recipe as given performs eigenanalysis, and subsequent orthogonalization, in exact values on an n-by-n matrix; this can be time- and resource-consuming for anything but the smallest of graphs. In practice, converting to floating-point values early on proves prudent. One can also work realization-by-realization: compute the eigenvalues first, then, for a given value, retrieve the eigenvectors via `NullSpace[adjMat - val * IdentityMatrix[n]]`, orthogonalizing an eigenspace at a time.[6] An in-between strategy would be to use `Eigensystem[]`, but to assemble the eigenmatrices prior to orthogonalization, which would proceed matrix-by-matrix.

Some concluding notes:

- Because a spectral realization (or its sub-dimensional embedding) of a graph requires no translational component in the isometry corresponding to a given automorphism, it implicitly defines a *representation* of the graph's automorphism group (or a subgroup thereof) by a group of *linear* isometries. This author is not familiar enough with representation theory to expound on the significance of this connection.

- When the graph G is *d-regular* —that is, when all vertices have degree d— then $[G]$ admits a one-dimensional d-eigenspace with the "all 1s" basis vector, $\mathbf{1}$. (The corresponding spectral realization —which we call the "dot"— collapses the entire graph into a single point.) The orthogonality of $[G]$'s eigenspaces then implies that $[\widehat{G}]\mathbf{1} = \mathbf{0}$ for any other spectral realization $\widehat{G}$; consequently, each non-dot spectral realization has its center at the origin.

  The skeleta of many familiar polyhedra —in particular, members of the Platonic and Archimedean families— have this property.

- Although each spectral realization is independent of any choice of orthogonal basis of eigenvectors for the adjacency matrix, (obviously) the lower-dimensional realizations provided Theorem 2 depend entirely upon the chosen basis; that is, there is no apparent "natural" projection.

  Arbitrary eigenvector orthogonalization —as is done in the *Mathematica* recipe— is unlikely to provide projected realizations with particularly "pretty" coordinates. Question: Is there a projection (which we might call "natural") that maximizes coordinate beauty?[7]

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[6] The *Mathematica*-based Demonstrations mentioned before use this strategy, in order to optimize interactivity with the viewer. On-the-fly nullspace computations (using floating point numbers) tend to be fairly speedy.

[7] I ask because, as you will see in the coming section, I have generated *hundreds* of graph realizations whose coordinates might as well be random. (Of course, they *aren't* random.) Going case-by-case to find the pretty coordinates "by hand" is far too daunting a task to undertake.

4. Examples

The remainder[8] of this paper comprises a survey the spectral realizations of various familiar graphs: from polygons (which arise from n-cycles), to (the skeleta of) the uniform polyhedra and (so far) some of their duals, to (the skeleta of) regular polytopes in four or more dimensions.

> **This survey is a work in progress.** Some analysis of complicated figures is rudimentary and unverified, and should not (yet) be taken as definitive. The reader is invited to submit additional (and, if necessary, correctional) information.

4.1. **Polygons (n-Cycles).** A polygon's skeleton is an n-cycle. Labeling vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ such that v_i is adjacent to v_{i-1} and v_{i+1} (with index arithmetic performed modulo n), we have that the (i,j)-th element in the adjacency matrix is 1 if and only if $j = i \pm 1 \mod n$.

$$\begin{bmatrix} 0 & 1 & 0 & & 0 & 0 & 1 \\ 1 & 0 & 1 & & 0 & 0 & 0 \\ 0 & 1 & 0 & \ddots & 0 & 0 & 0 \\ & & \ddots & \ddots & & & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & & 1 & 0 & 1 \\ 1 & 0 & 0 & & 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues of this matrix are $\lambda_k := 2\cos(2k\pi/n)$ for $k = 0, 1, 2, \ldots, \lfloor n/2 \rfloor$. The spectral λ_k-realization has coordinate matrix of the form

$$\mathbf{\Lambda}_k := c_k \left[\quad \cos\left(\frac{2\pi k}{n}(i-j)\right) \quad \right], \text{where } c_k = \begin{cases} 1/n, & k = 0, n/2 \\ 2/n, & \text{otherwise} \end{cases}$$

Each of these realizations has dimension at most 2, and has an isometric image in $\mathbb{R}^1$ or $\mathbb{R}^2$ with coordinate matrix of the form

$$\sqrt{\tfrac{1}{n}} \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \end{bmatrix} \qquad \text{for } k = 0 \ (\lambda_k = 2)$$

$$\sqrt{\tfrac{1}{n}} \begin{bmatrix} 1 & -1 & 1 & \ldots & -1 \end{bmatrix} \qquad \text{for } k = n/2 \ (\lambda_k = -2)$$

$$\sqrt{\tfrac{2}{n}} \begin{bmatrix} 1 & \cos(2\pi k/n) & \cos(4\pi k/n) & \ldots & \cos(2\pi k(n-1)/n) \\ 0 & \sin(2\pi k/n) & \sin(4\pi k/n) & \ldots & \sin(2\pi k(n-1)/n) \end{bmatrix} \qquad \text{otherwise}$$

[8] To the extent that almost 95% of something counts as a "remainder".

The table records the following information

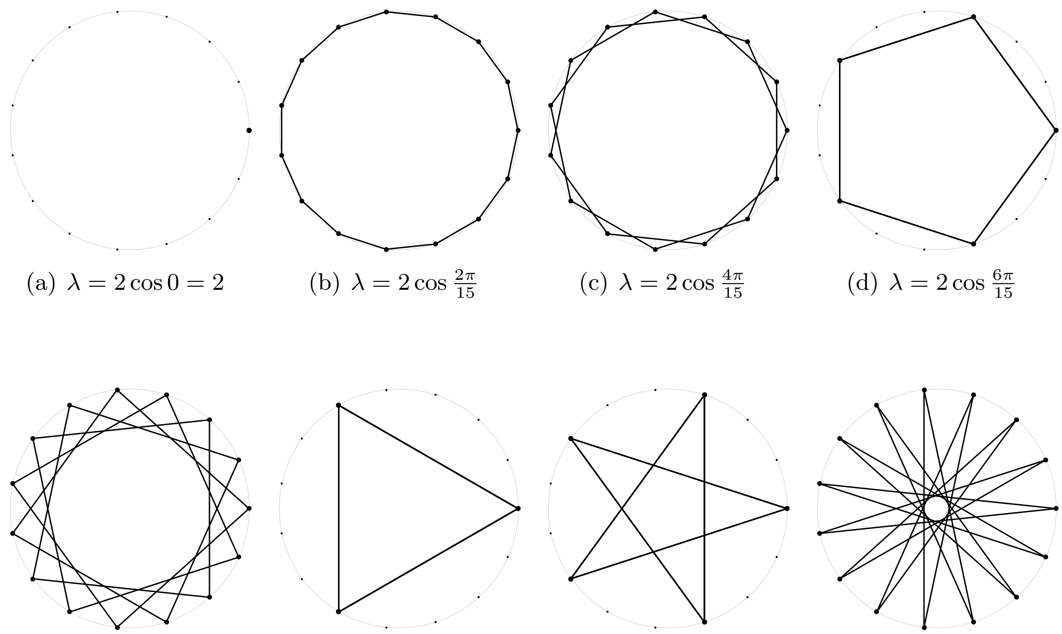
- λ. The eigenvalue of the corresponding eigenspace; this is also the *eigenic* scale factor of the realization. (Also given is the index k corresponding to the use in the above equations.)
- **Dim[ension]** of the eigenspace. In all cases but the dot, this is equal to the geometric dimension of the realization. (A dot is geometrically 0-dimensional, but appears as the realization of the 1-dimensional eigenspace spanned by the "all 1s" vector.)
- **#V[ertices] (n)**. The number of "coalesced" vertices. A *faithful* realization has n vertices.
- **Description**. The name of, or other notes about, the apparent figure.
- **Schläfli [Symbol]** $\{n/k\}$. The first number (n) counts the vertices; the second number (k, the "density" of the polygon) defines the arc —via $2\pi k/n$— travelled along a circumscribed unit circle from one vertex to the next during an n-step journey.[9] When $k = 0$, there is no travel, so that all vertices coincide, giving the dot realization.

n-gon (n-cycle)

λ	(k)	Dim	#V: n	Description	Schläfli
0	(0)	1	1	dot	$\{n/0\}$
$2\cos(2\pi/n)$	(1)	$\begin{cases} 1, & n = 2 \\ 2, & n \neq 2 \end{cases}$	n	convex regular	$\{n\}$ or $\{n/1\}$
$2\cos(2k\pi/n)$	(k)	$\begin{cases} 1, & k = n/2 \\ 2, & k \neq n/2 \end{cases}$	$\mathrm{lcm}(n,k)/k$	starry/multi-traced $\{n'/k'\}$ $n' = n/f,\ k' = k/f,$ $f = \mathrm{gcf}(n,k)$	$\{n/k\}$
-2	$(n/2)$	1	2	stick	$\{n/(n/2)\}$

Figure 3 shows the spectral realizations of the 15-cycle.

[9] Our usage follows Grünbaum and others, so that $\{6/2\}$ is a "multiply traced" triangle; this differs from the usage described in [4], which treats $\{6/2\}$ as a compound of two triangles.



(a) $\lambda = 2\cos 0 = 2$ (b) $\lambda = 2\cos\frac{2\pi}{15}$ (c) $\lambda = 2\cos\frac{4\pi}{15}$ (d) $\lambda = 2\cos\frac{6\pi}{15}$

(e) $\lambda = 2\cos\frac{8\pi}{15}$ (f) $\lambda = 2\cos\frac{10\pi}{15}$ (g) $\lambda = 2\cos\frac{12\pi}{15}$ (h) $\lambda = 2\cos\frac{14\pi}{15}$

Figure 3. The spectral realizations of the 15-cycle. Note that (a), (c), (e), and (h) are faithful; and (a), (d), (f), and (g) are "multiply-traced".

4.2. **Polyhedra.** Here, we investigate realizations of various highly-symmetric polyhedra. Note that, when we speak of the realization of a *polyhedron*, we mean more precisely, a realization of the vertex-and-edge *skeleton* of that polyhedron. We can consider a face defined abstractly by a cycle of edges on its "boundary", but we cannot, in general, give a suitable geometric interpretation of any region *bounded by* such a boundary, which may be (extremely) non-planar.

That being said, the organization of our data takes (spectrally irrelevant) face structure into account. For instance, the triangular-faced icosahedron (uniform polyhedron U_{22}) and the pentagonal-faced great dodecahedron (U_{35}) have isomorphic skeleta, which give rise to identical spectral realizations; however, we list the realization family of U_{22} separately from that of U_{35} (see Figures 21 and 33) in order that we can highlight the different face-cycles in each case. When polyhedra are isomorphic even in face structure —as with the icosahedron and great icosahedron (U_{53})— they are listed together, often appearing as spectral counterparts of each other.

The tables record the following information

- λ. The eigenvalue of the associated eigenspace of the adjacency matrix, and the *eigenic* scale factor of the realization.
- **Dim[ension]** of the eigenspace. In all cases but the dot, this is equal to the dimension of the associate eigenspace. (A dot is geometrically 0-dimensional, but appears as the realization of the 1-dimensional eigenspace spanned by the "all 1s" vector.)

- **Min[imal] Poly[nomial]** of the eigenvalue. The rows of the tables are grouped by these polynomials. (Note that realizations in a group necessarily have the same dimension.)

- **#V[ertices] (n).** The number of vertices the "apparent figure" —counting coincident vertices as individual elements— with n being the number of vertices in a *faithful* realization.

- **V[ertex] Fig[ure] ($a.b.c.\dots$).** The cyclic sequence of faces meeting a vertex, *without* counting coincident vertices as individual elements.[10] The sequence given in the column heading is for easy reference, and gives only the number of vertices in the faces, ignoring properties such as density and cycle-decomposition.

  Regular faces are listed as Schläfli(-like) symbols that, in non-convex realizations, sometimes denote "retrograde" tracing;[11] also, for dot realizations, the symbol $(a.b.c)/0$ is shorthand for $\frac{a}{0}.\frac{b}{0}.\frac{c}{0}$, a vertex figure consisting of dot-polygons.

  Note: The Schläfli notation is slowly being replaced by "cycle-decomposition notation", with component circumradii based on a polyhedral circumradius of 1. (See the Appendix at the end of this paper.) Unfortunately, CDN has no facility for denoting "retrograde" tracing.

- **Description**. Information, usually about the "apparent" figure.

  Some descriptions identify same-dimensional realizations that share vertices: *covert* (or, simply, *co*) realizations share complete sets of vertices; *subvert* (or *sub*) and *supvert* (or *sup*) realizations are such that the former's vertices form a subset of the latter's. (In many cases, the supvert realization's vertex set consists of the subvert realizations vertices and the reflections of those vertices through the origin.) Vertex family categorization is not yet complete (or necessarily reliable).

- $(r_0, r_1, \dots)$. (*Distance regular* graphs only.) Values such that the coordinate matrix of the spectral realization equals the sum $r_0\mathbf{A}_0 + r_1\mathbf{A}_1 + \cdots$, where $\mathbf{A}_d$ is the "distance-d adjacency matrix" (with d at most the combinatorial diameter of the graph) whose (i, j)-th entry is 1 if and only if the shortest path joining vertex i to vertex j has combinatorial length d. (By convention, $\mathbf{A}_0 = \mathbf{I}$.)

[10] An "apparent vertex figure" would likely be a useful addition to our data, but this isn't it.

[11] For instance, a $\{5/3\}$ pentagon is a $\{5/2\}$ pentagon traced "the other way around".

4.2.1. *The Uniform Polyhedra.* **This data is preliminary and incomplete.**

This subsection documents the spectral realizations of *uniform polyhedra*, defined by having regular polygonal faces and identical polyhedron vertices.[5] The data is organized by the standard indices, from U_1 (tetrahedron) to U_{75} (great dirhombicosidodecahedron), along with the infinite family of prismatic figures U_{76} through U_{80}. The Wythoff symbol [6] for each polyhedron is also given.

Uniform polyhedra are inherently harmonious realizations of their skeleta, but not all are eigenic. In many cases, a skelton's spectral family includes a "pseudo-uniform" representative evocative of a classical form, but having non-regular faces; in the tabulated data, such a realization is indicated by a tilde-topped symbol (e.g., $\tilde{U}_2$, suggesting an *approximate* U_2) and its non-regular faces (unless represented in cycle-decomposition notation (see Appendix)) by tilde-topped numbers in its vertex figure (e.g., $3.\tilde{6}.\tilde{6}$). In a handful of instances, the familiar realization is in fact eigenic, but not spectral, being a 3-dimensional projection of a higher-dimensional form; in the tables, the high-dimensional representative is indicated with a hat-topped symbol (e.g., $\hat{U}_{30}$, suggesting a *super-dimensional* U_{30} "up there"). The following table summarizes how the various uniform polyhedra fit (or don't fit) into the spectral families of their skeleta.[12]

U_1	U_4	$\cong$	U_5	U_3	$\cong$	U_7	$\cong$	U_{15}				U_k	:	19
$\tilde{U}_2$				$\tilde{U}_{27}$	$\cong$	$\tilde{U}_{33}$	$\cong$	$\tilde{U}_{39}$				$\tilde{U}_k$	:	27
U_6	$\tilde{U}_9$	$\equiv$	$\tilde{U}_{19}$	$\hat{U}_{30}$	$\cong$	$\hat{U}_{41}$	$\cong$	$\hat{U}_{47}*$				$\hat{U}_k$	:	7
$\tilde{U}_8$	$\tilde{U}_{11}$	$\equiv$	(U_{20})	(U_{38})	$\cong$	(U_{44})	$\cong$	(U_{56})				(U_k)	:	27
$\tilde{U}_{12}$	U_{23}	$\equiv$	U_{52}	$\hat{U}_{36}$	$\equiv$	$\hat{U}_{62}$	$\equiv$	$\hat{U}_{65}$				All	:	80
(U_{16})	$\tilde{U}_{25}$	$\equiv$	$\tilde{U}_{55}$	$\tilde{U}_{57}$	$\equiv$	$\tilde{U}_{69}$	$\equiv$	$\tilde{U}_{74}$						
$\tilde{U}_{29}$	$\tilde{U}_{26}$	$\equiv$	(U_{66})	(U_{61})	$\equiv$	(U_{67})	$\equiv$	(U_{73})						
(U_{45})	$\tilde{U}_{28}$	$\equiv$	(U_{68})	(U_{77})	$\equiv$	(U_{79})	$\equiv$	(U_{80})						
(U_{46})	$\tilde{U}_{32}$	$\equiv$	$\tilde{U}_{72}$	U_{22}	$\equiv$	U_{53}	$\cong$	U_{34}	$\equiv$	U_{35}				
(U_{59})	$\tilde{U}_{37}$	$\equiv$	(U_{58})	$\tilde{U}_{10}$	$\equiv$	(U_{17})	$\cong$	$\tilde{U}_{13}$	$\equiv$	(U_{14})	$\cong$	$\tilde{U}_{18}$	$\equiv$	(U_{21})
(U_{64})	(U_{40})	$\equiv$	(U_{60})	U_{24}	$\equiv$	U_{54}	$\cong$	U_{49}	$\equiv$	U_{71}	$\cong$	U_{51}	$\equiv$	U_{70}
$\hat{U}_{75}$	(U_{76})	$\equiv$	(U_{78})	$\tilde{U}_{31}$	$\equiv$	$\tilde{U}_{48}$	$\cong$	(U_{42})	$\equiv$	$\tilde{U}_{43}$	$\cong$	$\tilde{U}_{50}$	$\equiv$	(U_{63})

U_k : classical form is spectral	$\tilde{U}_k$: pseudo-classical form is spectral
$\hat{U}_k$: classical form is eigenic projection	(U_k): classical form is far from spectral
$\cong$: skeleta are isomorphic	$\equiv$: skeleta and face structures are isomorphic

FIGURE 4. The 80 uniform polyhedra arranged into 35 classes having isomorphic skeleta.

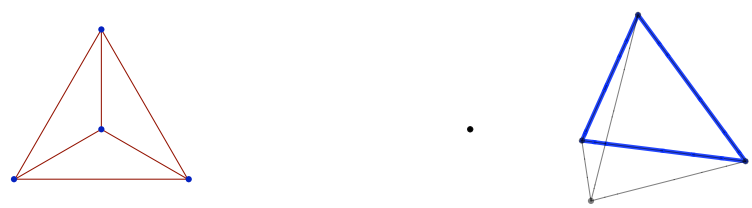
The figures depict a polyhedron's skeleton and its the "low" (≤ 3)-dimensional realizations. In each case, representative face cycles have been highlighted. When a polyhedron's "classical" form does not appear in the spectral family, that form is shown separately.

[12]Of note: Except for the cube, each "classical form is spectral" figure has the property that every edge borders a triangular or pentagonal face, which *must* be regular, given the required rotational symmetry. (The cube gets into the club through connections: its geometric dual —the octahedron— is all triangles.) The symmetry aspect is essential: pentagonal anti-prisms have this edge property but don't exhibit three-fold symmetry in their triangular faces; their classical forms are not spectral.

U_1 Tetrahedron $[3|2,3]$

λ	Dim	Min Poly	#V: 4	V Fig: 3^3	Description	$(r_0, r_1, \dots)$
3	1	$x-3$	1	$[3:1,0]^3$	dot	$\frac{1}{4}(1,1)$
-1	3	$x+1$	4	$[3:\alpha,\beta]^3$	U_1	$\frac{1}{4}(3,-1)$

$\alpha = 1/3, \beta = 2\sqrt{2}/3 \approx 0.9428$



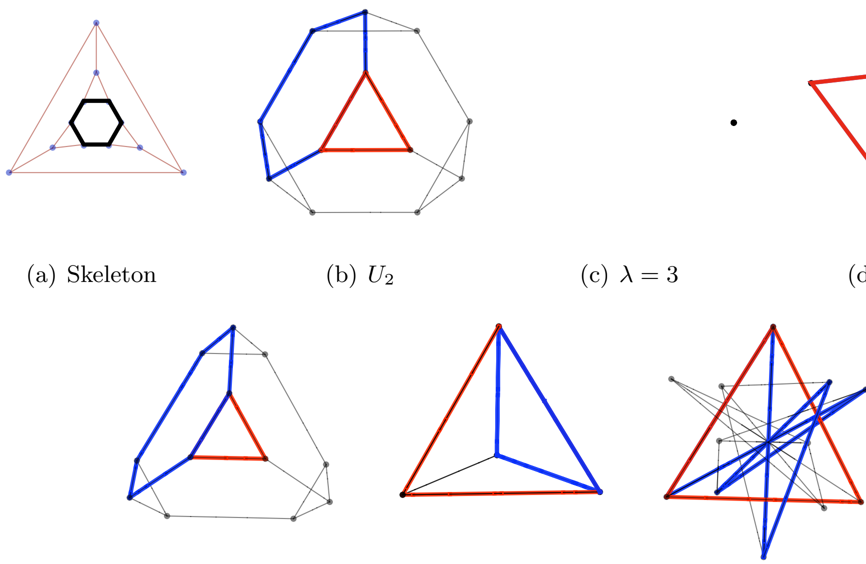
(a) Skeleton (b) $\lambda = 3$ (c) $\lambda = -1 : U_1$

FIGURE 5. The U_1 skeleton (a) and its low-dimensional spectral realizations.

U_2 Truncated Tetrahedron $[2, 3|3]$

λ	Dim	Min Poly	#V: 12	V Fig: $6^2.3$	Description
3	1	$x-3$	1	$[6:1,0,0,0]^2.[3:1,0]$	dot
0	2	x	3	$[6:0,\alpha,\beta,0]^2.[3:1,0]$	triangle
2	3	$x-2$	12	$[6:\gamma,\alpha,\delta,0]^2.[3:\epsilon,\zeta]$	$\tilde{U}_2$
-1	3	$x+1$	4	$[6:\zeta,0,\epsilon,0]^2.[3:\zeta,\epsilon]$	tetrahedron
-2	3	$x+2$	12	$[6:0,\eta,\beta,\theta]^2.[3:0,1]$	4-tri. compoundex

$\alpha = \sqrt{3}/2 \approx 0.8660, \beta = 1/2, \gamma = \sqrt{2}/3 \approx 0.4714, \delta = 1/6, \epsilon = 2\sqrt{2}/3 \approx 0.9428, \zeta = 1/3$
$\eta = \sqrt{3}/6 \approx 0.2886, \theta = \sqrt{6}/3 \approx 0.8164$



(a) Skeleton (b) U_2 (c) $\lambda = 3$ (d) $\lambda = 0$

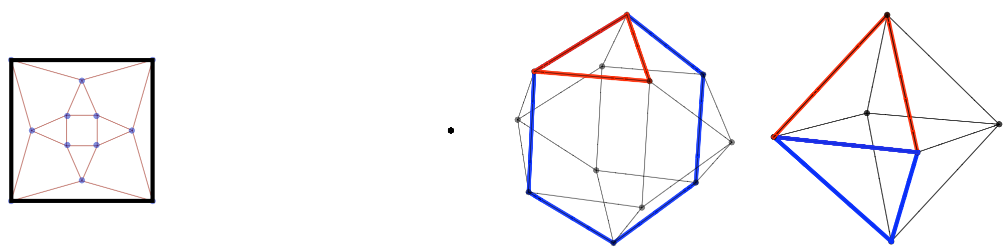
(e) $\lambda = 2 : \tilde{U}_2$ (f) $\lambda = -1$ (g) $\lambda = -2$

FIGURE 6. The U_2 skeleton (a), its classical realization (b), and its low-dimensional spectral realizations. (The 2-realization is only pseudo-classical: the hexagonal faces are not regular.)

U_3 Octahemioctahedron $[3/2, 3|3]$ skel $\cong U_7, U_{15}$

λ	Dim	Min Poly	#V: 12	V Fig: $(6.3)^2$	Description
4	1	$x-4$	1	$([6:1,0,0,0].[3:1,0])^2$	dot
2	3	$x-2$	12	$([6:0,1,0,0].[3:\alpha,\beta])^2$	U_3
0	3	x	6	$([6:\beta,0,\alpha,0].[3:\beta,\alpha])^2$	octahedron
-2	5	$x+2$	12	$([6:\gamma,\delta,\delta].[3:0,1])^2$	$-$

$\alpha = \sqrt{6}/3 \approx 0.8164, \beta = \sqrt{3}/3 \approx 0.5773, \gamma = \sqrt{5}/5 \approx 0.4472, \delta = \sqrt{10}/5 \approx 0.6324$



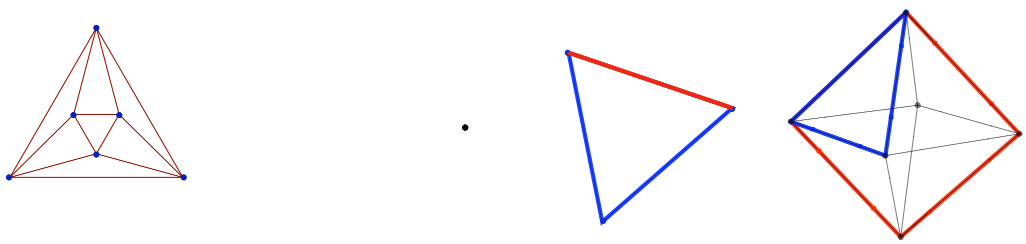
(a) Skeleton (b) $\lambda = 4$ (c) $\lambda = 2 : U_3$ (d) $\lambda = 0$

FIGURE 7. The U_3 skeleton (a) and its low-dimensional spectral realizations.

U_4 Tetrahemihexahedron $[3/2, 3|2]$ skel $\cong U_5$

λ	Dim	Min Poly	#V: 6	V Fig: $(4.3)^2$	Description	$(r_0, r_1, \dots)$
4	1	$x-4$	1	$([4:1,0,0].[3:1,0])^2$	dot	$\frac{1}{6}(1,1,1)$
-2	2	$x+2$	3	$([4:\alpha,0,\beta].[3:0,1])^2$	triangle	$\frac{1}{6}(2,-1,2)$
0	3	x	6	$([4:0,1,0].[3:\gamma,\delta,0])^2$	U_4	$\frac{1}{2}(1,0,-1)$

$\alpha = \frac{1}{2}, \beta = \sqrt{3}/2 \approx 0.8660, \gamma = \sqrt{3}/3 \approx 0.5773, \delta = \sqrt{6}/3 \approx 0.8164$

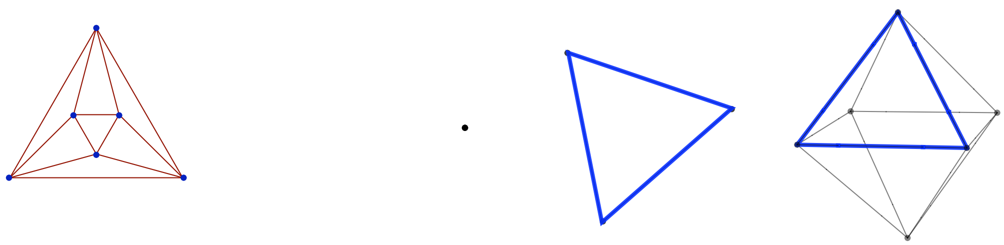


(a) Skeleton (b) $\lambda = 4$ (c) $\lambda = -2$ (d) $\lambda = 0 : U_4$

FIGURE 8. The U_4 skeleton (a) and its low-dimensional spectral realizations.

			U_5	Octahedron	$[4\|2,3]$	skel $\cong U_4$
λ	Dim	Min Poly	#V: 6	V Fig: 3^4	Description	$(r_0, r_1, \dots)$
4	1	$x-4$	1	$[3:1,0]^4$	dot	$\frac{1}{6}\,(1,1,1)$
-2	2	$x+2$	3	$[3:0,1]^4$	triangle	$\frac{1}{6}\,(2,-1,2)$
0	3	x	6	$[3:\alpha,\beta]^4$	U_5	$\frac{1}{2}\,(1,0,-1)$

$\alpha = \sqrt{3}/3 \approx 0.5773, \beta = \sqrt{6}/3 \approx 0.8164$



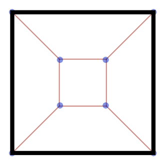
(a) Skeleton (b) $\lambda = 4$ (c) $\lambda = -2$ (d) $\lambda = 0 : U_5$

FIGURE 9. The U_5 skeleton (a) and its low-dimensional spectral realizations.

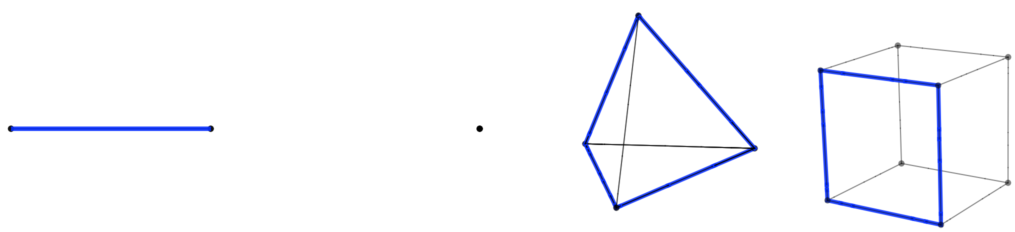
U_6 Cube $[3|2,4]$

λ	Dim	Min Poly	#V: 8	V Fig: 4^3	Description	$(r_0, r_1, \dots)$
-3	1	$x+3$	2	$[4:0,0,1]^3$	stick	$\frac{1}{8}\,(1,-1,1,-1)$
3	1	$x-3$	1	$[4:1,0,0]^3$	dot	$\frac{1}{8}\,(1,1,1,1)$
-1	3	$x+1$	4	$[4:\alpha,\beta,0]^3$	$\pi\{3,3\}^*$ (subvert)	$\frac{1}{8}\,(3,-1,-1,3)$
1	3	$x-1$	8	$[4:0,\beta,\alpha]^3$	U_6 (supvert)	$\frac{1}{8}\,(3,1,-1,-3)$

$\alpha = \sqrt{3}/3 \approx 0.5773, \beta = \sqrt{6}/3 \approx 0.8164$
* The Petrie polyhedron based on the tetrahedron.



(a) Skeleton



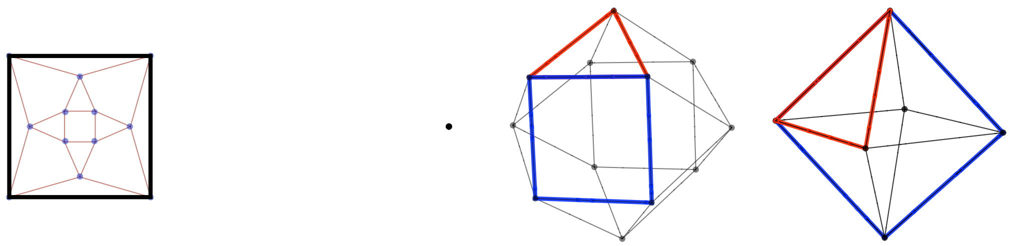
(b) $\lambda = -3$ (c) $\lambda = 3$ (d) $\lambda = -1$ (e) $\lambda = 1 : U_6$

FIGURE 10. The U_6 skeleton (a) and its low-dimensional spectral realizations.

U_7 Cuboctahedron $[2|3,4]$ skel $\cong U_3, U_{15}$

λ	Dim	Min Poly	#V: 12	V Fig: $(3.4)^2$	Description
4	1	$x-4$	1	$([3:1,0].[4:1,0,0])^2$	dot
2	3	$x-2$	12	$([3:\alpha,\beta].[4:\gamma,\gamma,0])^2$	U_7
0	3	x	6	$([3:\beta,\alpha].[4:0,1,0])^2$	U_4
-2	5	$x+2$	12	$([3:0,1].[4:\delta,\epsilon,\zeta])^2$	-

$\alpha = 0.8164, \beta = 0.5773, \gamma = 0.7071, \delta = 0.3162, \epsilon = 0.5477, \zeta = 0.7745$



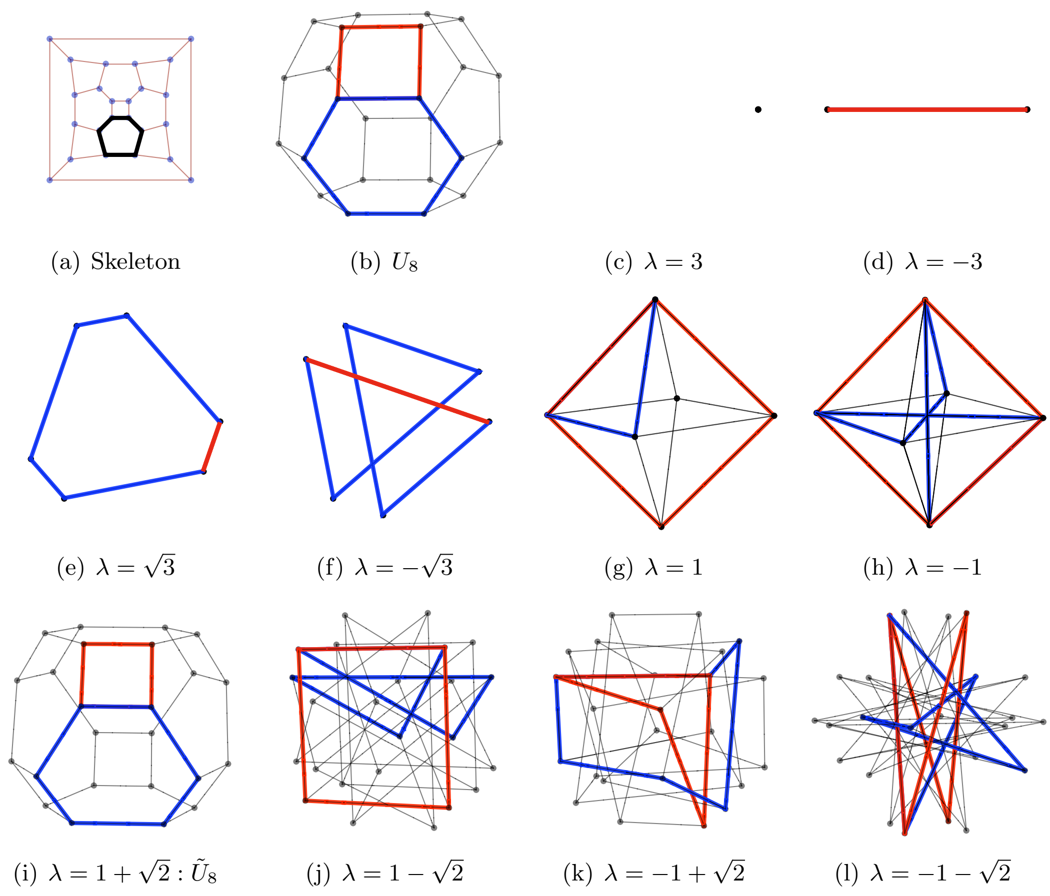
(a) Skeleton (b) $\lambda = 4$ (c) $\lambda = 2 : U_7$ (d) $\lambda = 0$

FIGURE 11. The U_7 skeleton (a) and its low-dimensional spectral realizations.

U_8 Truncated Octahedron $[2,4|3]$

λ	Dim	Min Poly	#V: 24	V Fig: $6^2.4$	Description
-3	1	$x+3$	2	$[6:0,0,0,1]^2.[4:0,0,1]$	stick
3	1	$x-3$	1	$[6:1,0,0,0]^2.[4:1,0,0]$	dot
$\sqrt{3}\approx 1.732$ $-\sqrt{3}\approx -1.732$	2	x^2-3	6	$[6:0,\alpha,\beta,0]^2.[4:\alpha,0,\beta]$ $[6:0,\beta,\alpha,0]^2.[4:\beta,0,\alpha]$	- (co I) - (co I)
1	3	$x-1$	6	$[6:\nu,\xi,\rho,0]^2.[4:0,1,0]$	- (co II)
-1	3	$x+1$	6	$[6:0,\rho,\xi,\nu]^2.[4:0,1,0]$	- (co II)
$1+\sqrt{2}\approx 2.414$ $1-\sqrt{2}\approx -0.414$	3	x^2-2x-1	24	$[6:\gamma,\delta,\epsilon,0]^2.[4:\zeta,\eta,0]$ $[6:\mu,\kappa,\theta,0]^2.[4:\eta,\zeta,0]$	$\tilde{U}_8$ (co III) - (co III)
$-1+\sqrt{2}\approx 0.414$ $-1-\sqrt{2}\approx -2.414$	3	x^2+2x-1	24	$[6:0,\theta,\kappa,\mu]^2.[4:0,\zeta,\eta]$ $[6:0,\epsilon,\delta,\gamma]^2.[4:0,\eta,\zeta]$	- (co III) - (co III)

$\alpha=0.9659, \beta=0.2588, \gamma=0.7543, \delta=0.6532, \epsilon=0.0647, \zeta=0.9238,$
$\eta=0.3826, \theta=0.9105, \kappa=0.2705, \mu=0.3124, \nu=0.5773, \xi=0.7071, \rho=0.4082$



(a) Skeleton (b) U_8 (c) $\lambda=3$ (d) $\lambda=-3$

(e) $\lambda=\sqrt{3}$ (f) $\lambda=-\sqrt{3}$ (g) $\lambda=1$ (h) $\lambda=-1$

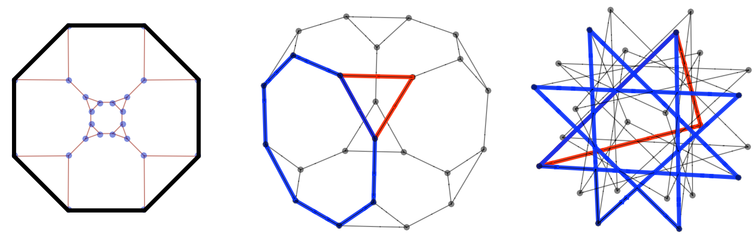
(i) $\lambda=1+\sqrt{2}:\tilde{U}_8$ (j) $\lambda=1-\sqrt{2}$ (k) $\lambda=-1+\sqrt{2}$ (l) $\lambda=-1-\sqrt{2}$

FIGURE 12. The U_8 skeleton (a), its classical realization (b), and its low-dimensional spectral realizations. (The $(1+\sqrt{2})$-realization is only pseudo-classical: the hexagonal faces are not regular.)

U_9	Truncated Cube	$[2, 3\|4]$
U_{19}	Stellated Truncated Hexahedron	$[2, 3\|4/3]$

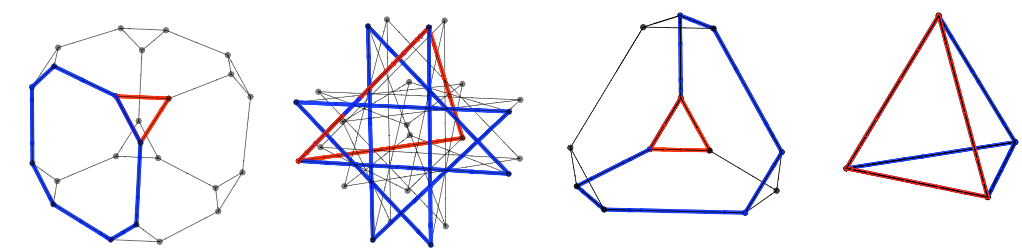
λ	Dim	Min Poly	#V: 24	V Fig: $8^2.3$	Description
3	1	$x-3$	1	$[8:1,0,0,0,0]^2.[3:1,0]$	dot
1	1	$x-1$	2	$[8:0,0,\frac{\alpha}{\alpha}0,0,0]^2.[3:1,0]$	stick
$\frac{1}{2}\left(1+\sqrt{17}\right) \approx 2.561$ $\frac{1}{2}\left(1-\sqrt{17}\right) \approx -1.561$	3	x^2-x-4	24	$[8:\beta,\gamma,0,\delta,0]^2.[3:\epsilon,\zeta]$ $[8:\eta,\theta,0,\kappa,0]^2.[3:\zeta,\epsilon]$	$\tilde{U_9}$ $\tilde{U}_{19}$
2	3	$x-2$	12	$[8:0,\mu,\frac{\nu}{\nu}0,\xi,0]^2.[3:\rho,\sigma]$	$\tilde{U_2}$
-1	3	$x+1$	4	$[8:0,\tau,\frac{\upsilon}{\upsilon}0,\phi,0]^2.[3:\sigma,\rho]$	U_1
-2	5	$x+2$	24	-	- (sup II)
0	5	x	12	-	- (sub II)

$\alpha = \sqrt{2}/2 \approx 0.7071, \beta \approx 0.6571, \gamma \approx 0.7483, \delta \approx 0.0894, \epsilon \approx 0.9719, \zeta \approx 0.2352$
$\eta \approx 0.2609, \theta \approx 0.1145, \kappa \approx 0.9585, \mu \approx 0.9105, \nu \approx 0.2886, \xi \approx 0.0647, \rho \approx 0.9428, \sigma = 1/3$
$\tau \approx 0.3124, \upsilon \approx 0.4082, \phi \approx 0.7543$



(a) Skeleton (b) U_9 (c) U_{19}

(d) $\lambda = 3$ (e) $\lambda = 1$

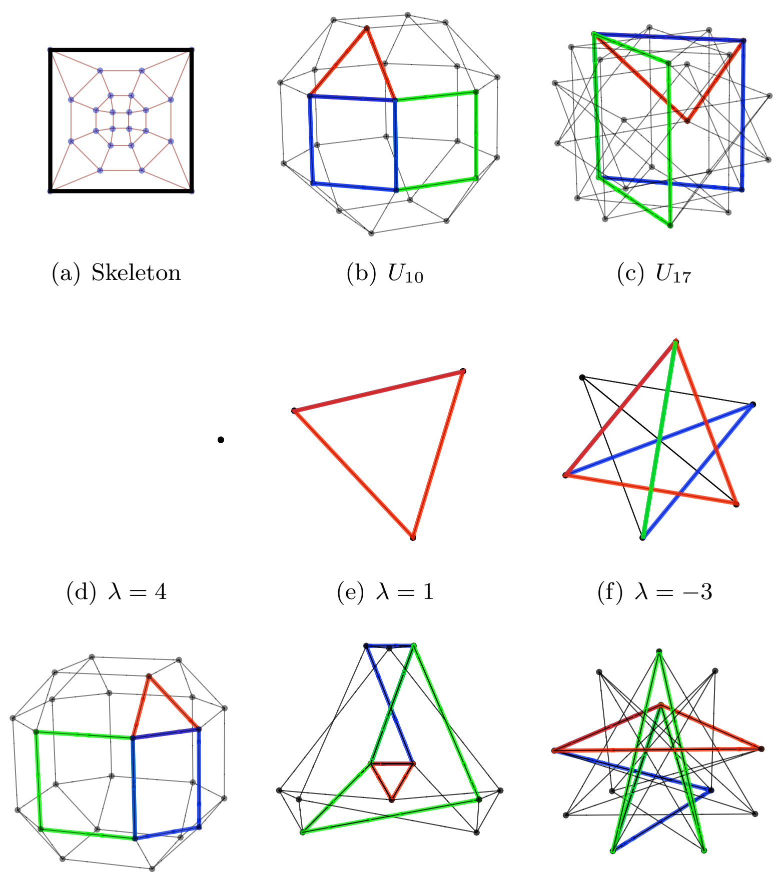


(f) $\lambda = \frac{1}{2}(1+\sqrt{17}) : \tilde{U}_9$ (g) $\lambda = \frac{1}{2}(1-\sqrt{17}) : \tilde{U}_{19}$ (h) $\lambda = 2$ (i) $\lambda = -1$

FIGURE 13. The U_9 / U_{19} skeleton (a), its classical realizations (b, c), and its low-dimensional spectral realizations. (The $\frac{1}{2}(1+\sqrt{17})$- and $\frac{1}{2}(1-\sqrt{17})$-realizations are only pseudo-classical: their octagonal faces are not regular.)

U_{10} Small Rhombicuboctahedron $[3, 4|2]$
U_{17} Uniform Great Rhombicuboctahedron $[3/2, 4|2]$ skel $\cong U_{13}, U_{14}, U_{18}, U_{21}$

λ	Dim	Min Poly	#V: 24	V Fig: 3.4^3	Description
4	1	$x-4$	1	$(3.4^3)/0$	dot
1	2	$x-1$	3	-	triangle (sub I)
-3	2	$x+3$	6	-	2-tri. compoundex (sup I)
3	3	$x-3$	24	$3.\tilde{4}.4.\tilde{4}$	$\tilde{U}_{10}$
$\frac{1}{2}\left(-1+\sqrt{17}\right) \approx 1.561$ $\frac{1}{2}\left(-1-\sqrt{17}\right) \approx -2.561$	3	x^2+x-4	12	- -	- -
0	4	x	8	-	-
-1	6	$x+1$	24	-	-



(a) Skeleton (b) U_{10} (c) U_{17}

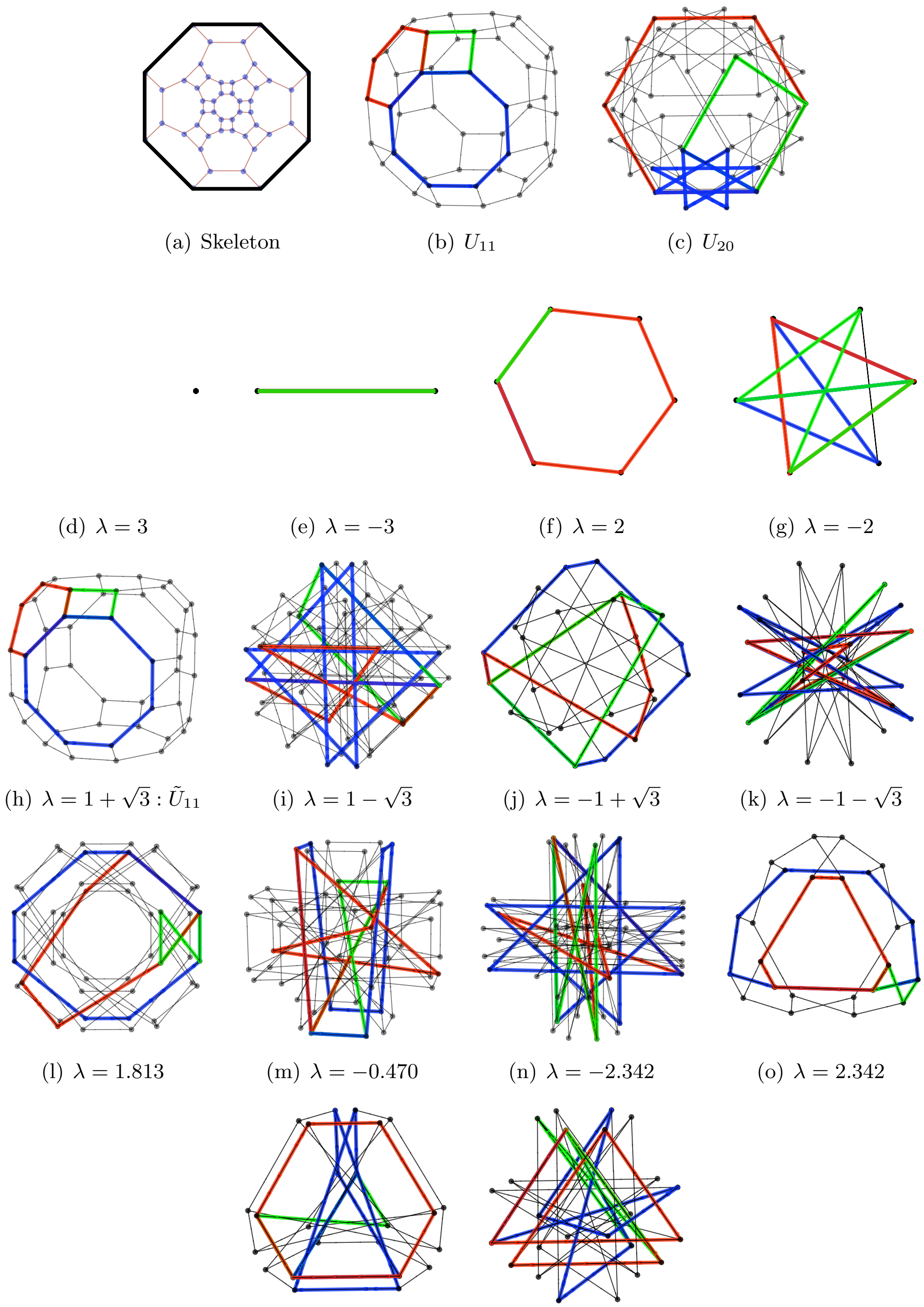
(d) $\lambda = 4$ (e) $\lambda = 1$ (f) $\lambda = -3$

(g) $\lambda = 3 : \tilde{U}_{10}$ (h) $\lambda = \frac{1}{2}(-1+\sqrt{17})$ (i) $\lambda = \frac{1}{2}(-1-\sqrt{17})$

FIGURE 14. The U_{10} / U_{17} skeleton (a), its classical realizations (b, c), and its low-dimensional spectral realizations. (The 3-realization is only pseudo-classical: the triangular faces have non-square neighbors.)

U_{11} Great Rhombicuboctahedron $[2, 3, 4|]$
U_{20} Great Truncated Cuboctahedron $[4/3, 2, 3|]$

λ	Dim	Min Poly	#V: 48	V Fig: 4.6.8	Description
3	1	$x-3$	1	(4.6.8)/0	dot
-3	1	$x+3$	2	-	stick
2	2	$x-2$	6	-	hexagon (co I)
-2	2	$x+2$	6	-	2-tri. compoundex (co I)
$1+\sqrt{3} \approx 2.732$ $1-\sqrt{3} \approx -0.732$	3	x^2-2x-2	48	$\tilde{4}.\tilde{6}.\tilde{8}$ -	$\tilde{U}_{11}$ (co II, sup II) - (co II)
$-1+\sqrt{3} \approx 0.732$ $-1-\sqrt{3} \approx -2.732$	3	x^2+2x-2	24	- -	- (co III, sub II) - (co III)
1.813 -0.470 -2.342	3	x^3+x^2-4x-2	48	- - -	- (sup IV) - (sup VI) - (sup III)
2.342 0.470 -1.813	3	x^3-x^2-4x+2	24	- - -	- (sub III) - (sub VI) - (sub IV)
1	4	$x-1$	8	-	- (covert II)
-1	4	$x+1$	8	-	- (covert II)
0	4	x	6	-	-



(a) Skeleton (b) U_{11} (c) U_{20}

(d) $\lambda = 3$ (e) $\lambda = -3$ (f) $\lambda = 2$ (g) $\lambda = -2$

(h) $\lambda = 1 + \sqrt{3} : \tilde{U}_{11}$ (i) $\lambda = 1 - \sqrt{3}$ (j) $\lambda = -1 + \sqrt{3}$ (k) $\lambda = -1 - \sqrt{3}$

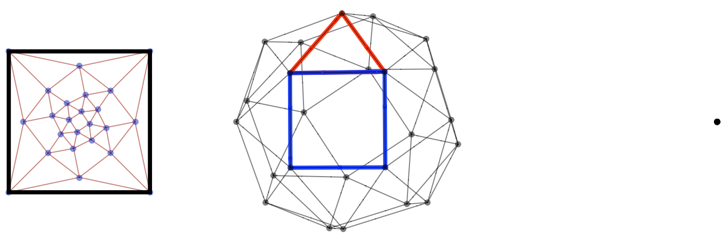
(l) $\lambda = 1.813$ (m) $\lambda = -0.470$ (n) $\lambda = -2.342$ (o) $\lambda = 2.342$

(p) $\lambda = 0.470$ (q) $\lambda = -1.813$

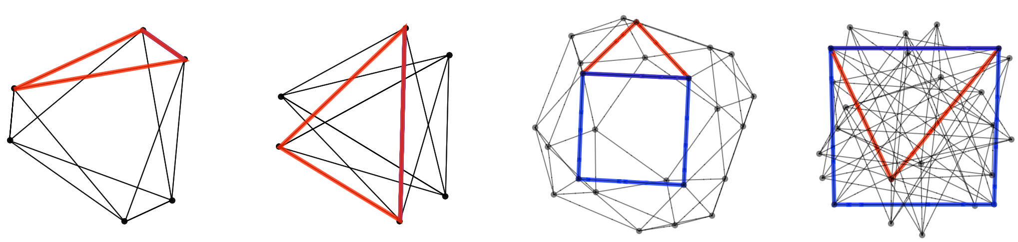
Figure 15. The U_{11} / U_{20} skeleton (a), its classical realizations (b,c), and its low-dimensional spectral realizations. (The $(1+\sqrt{3})$-realization is only pseudo-classical: none of its faces are regular.)

U_{12} Snub Cube $[|2,3,4]$

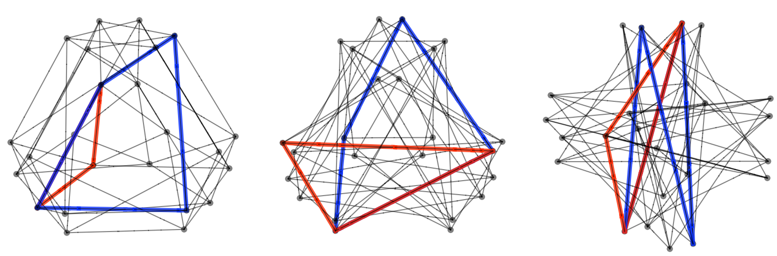
λ	Dim	Min Poly	#V: 24	V Fig: 4.3^4	Description
5	1	$x-5$	1	$(4.3^4)/0$	dot
$-1+\sqrt{3}\approx 0.732$ $-1-\sqrt{3}\approx -2.732$	2	x^2+2x-2	6	- -	- (co I) - (co I)
$1+\sqrt{7}\approx 3.645$ $1-\sqrt{7}\approx -1.645$	3	x^2-2x-6	24	$4.\tilde{3}.3.\tilde{3}.\tilde{3}$ -	$\tilde{U}_{12}$ -
1.813 -0.470 -2.342	3	x^3+x^2-4x-2	24	- - -	- - -
-1	4	$x+1$	8	-	-



(a) Skeleton (b) U_{12} (c) $\lambda = 5$



(d) $\lambda=-1+\sqrt{3}$ (e) $\lambda=-1-\sqrt{3}$ (f) $\lambda=1+\sqrt{7}:\tilde{U}_{12}$ (g) $\lambda=1-\sqrt{7}$

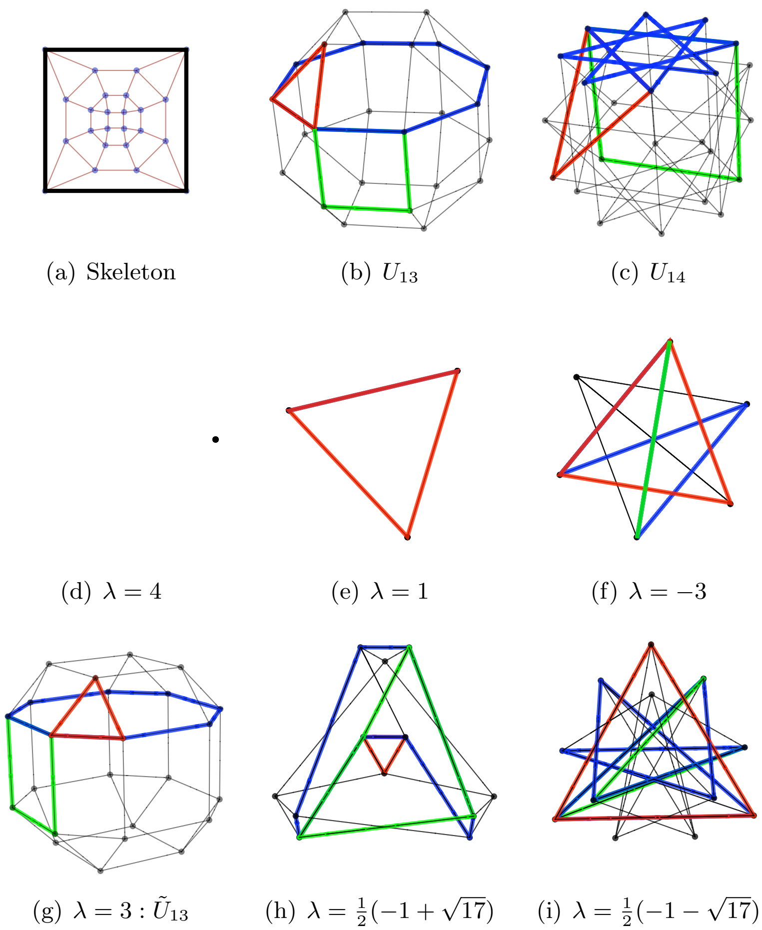


(h) $\lambda=1.813$ (i) $\lambda=-0.470$ (j) $\lambda=-2.342$

FIGURE 16. The U_{12} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations. (The $(1+\sqrt{7})$-realization is only pseudo-classical: the triangular neighbors of the square faces are not regular.)

U_{13} Small Cubicuboctahedron $[3/2, 4|4]$
U_{14} Great Cubicuboctahedron $[3, 4|4/3]$ skel $\cong U_{10}, U_{17}, U_{18}, U_{21}$

λ	Dim	Min Poly	#V: 24	V Fig: 4.8.3.8	Description
4	1	$x-4$	1	(4.8.3.8)/0	dot
1	2	$x-1$	3	-	triangle (sub I)
-3	2	$x+3$	6	-	2-tri. compoundex (sup I)
3	3	$x-3$	24	$4.\tilde{8}.3/2.\tilde{8}$	$\tilde{U}_{13}$
$\frac{1}{2}\left(-1+\sqrt{17}\right) \approx 1.561$ $\frac{1}{2}\left(-1-\sqrt{17}\right) \approx -2.561$	3	x^2+x-4	12	- -	- -
0	4	x	8	-	-
-1	6	$x+1$	24	-	-



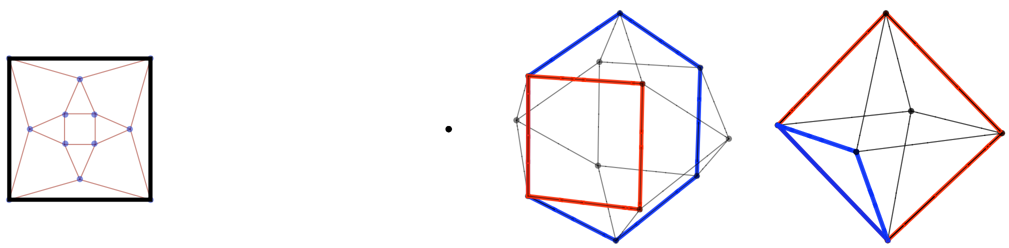
(a) Skeleton (b) U_{13} (c) U_{14}

(d) $\lambda = 4$ (e) $\lambda = 1$ (f) $\lambda = -3$

(g) $\lambda = 3 : \tilde{U}_{13}$ (h) $\lambda = \frac{1}{2}(-1+\sqrt{17})$ (i) $\lambda = \frac{1}{2}(-1-\sqrt{17})$

FIGURE 17. The U_{13} / U_{14} skeleton (a), its classical realizations (b, c), and its low-dimensional spectral realizations. (The 3-realization is only pseudo-classical: the octagonal faces are non-regular.)

U_{15}	Cubohemioctahedron	$[4/3, 4\|3]$	skel $\cong U_3, U_7$		
λ	Dim	Min Poly	#V: 12	V Fig: $(4.6)^2$	Description
4	1	$x-4$	1	$(4.6)^2/0$	dot
2	3	$x-2$	12	4.6.4/3.6	U_{15}
0	3	x	6	-	octahedron
-2	5	$x+2$	12	-	$x+2$

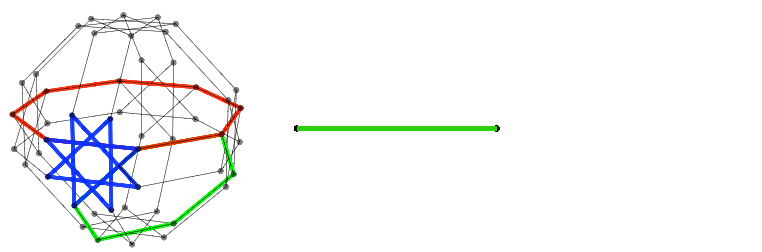


(a) Skeleton (b) $\lambda = 4$ (c) $\lambda = 2 : U_{15}$ (d) $\lambda = 0$

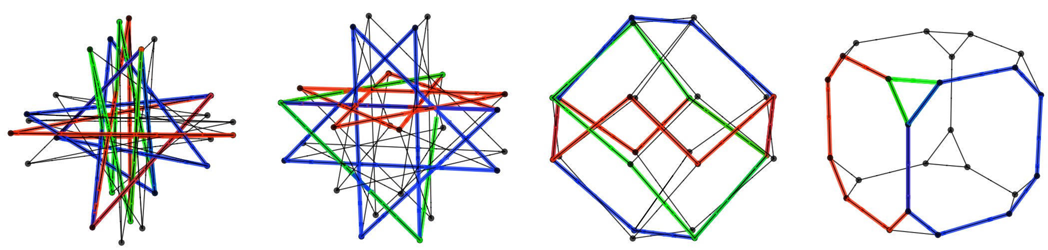
FIGURE 18. The U_{15} skeleton (a) and its low-dimensional spectral realizations.

U_{16} Cubitruncated Cuboctahedron $[4/3, 3, 4|]$

λ	Dim	Min Poly	#V: 48	V Fig: 6.8^2	Description
-3	1	$x+3$	2	-	stick
3	1	$x-3$	1	$(6.8^2)/0$	dot
-2.561 1.561	3	x^2+x-4	- -	- -	- -
-1.561 2.561	3	x^2-x-4	- -	- -	- -
-1	4	$x+1$	-	-	-
1	4	$x-1$	-	-	-
-2	8	$x+2$	-	-	-
2	8	$x-2$	-	-	-
0	10	x	-	-	-



(a) Skeleton (b) U_{16} (c) $\lambda = -3$ (d) $\lambda = 3$

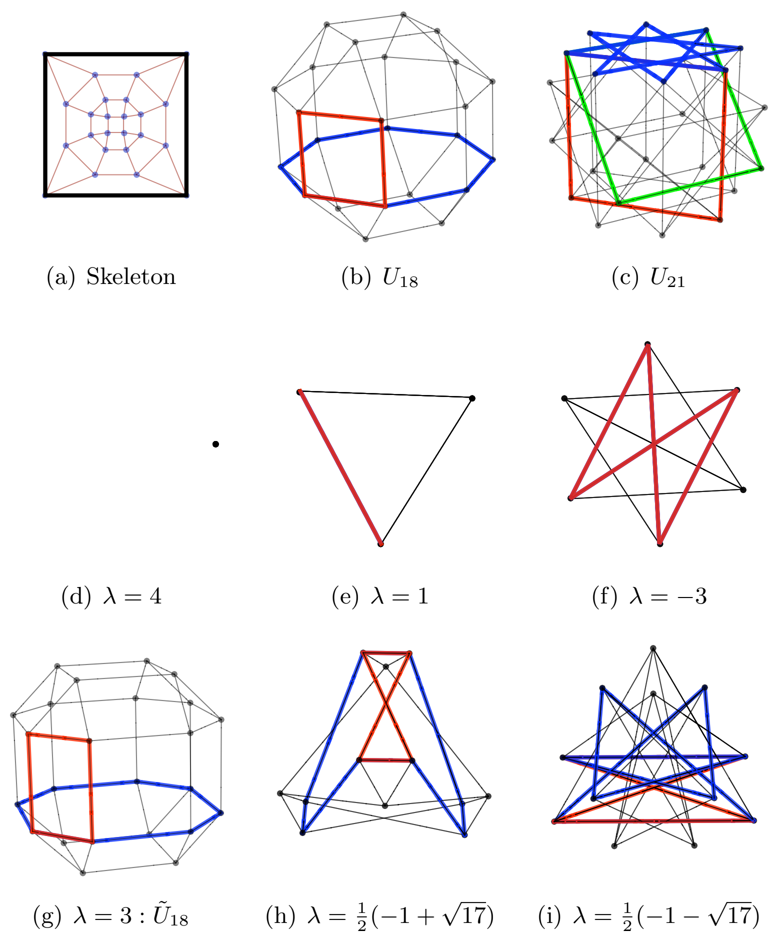


(e) $\lambda = -2.561$ (f) $\lambda = -1.561$ (g) $\lambda = 1.561$ (h) $\lambda = 2.561$

FIGURE 19. The U_{16} skeleton (a) and its low-dimensional spectral realizations.

U_{17} : see U_{10}

U_{18} Small Rhombihexahedron $[3/2, 2, 4\|]$ U_{21} Great Rhombihexahedron $[4/3, 3/2, 2\|]$ skel $\cong U_{10}, U_{17}, U_{13}, U_{14}$					
λ	Dim	Min Poly	#V: 24	V Fig: $(4.8)^2$	Description
4	1	$x-4$	1	$(4.8)^2/0$	dot
1	2	$x-1$	3	-	triangle (sub I)
-3	2	$x+3$	6	-	2-tri. cmpndx (sup I)
3	3	$x-3$	24	$\tilde{4}.\tilde{8}.\tilde{4}/3.\tilde{8}$	$\tilde{U}_{18}$
$\frac{1}{2}(-1+\sqrt{17}) \approx 1.561$ $\frac{1}{2}(-1-\sqrt{17}) \approx -2.561$	3	x^2+x-4	12	- -	- -
0	4	x	8	-	-
-1	6	$x+1$	24	-	-



(a) Skeleton (b) U_{18} (c) U_{21}

(d) $\lambda = 4$ (e) $\lambda = 1$ (f) $\lambda = -3$

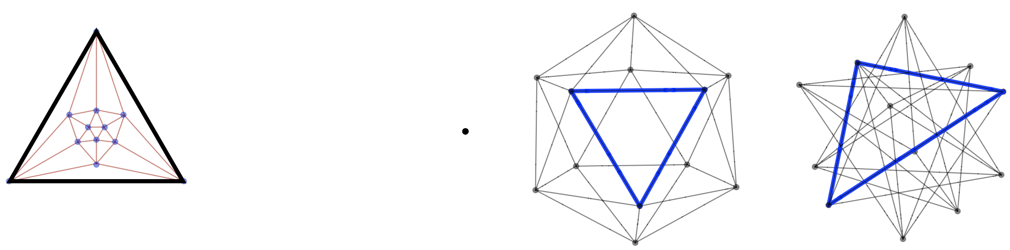
(g) $\lambda = 3 : \tilde{U}_{18}$ (h) $\lambda = \frac{1}{2}(-1+\sqrt{17})$ (i) $\lambda = \frac{1}{2}(-1-\sqrt{17})$

FIGURE 20. The U_{18} / U_{21} skeleton (a), its classical realizations (b, c), and its low-dimensional spectral realizations. (The 3-realization is only pseudo-classical: none of its faces are regular.)

U_{19} : see U_9
U_{20} : see U_{11}
U_{21} : see U_{18}

U_{22} Icosahedron $[5|2,3]$
U_{53} Great Icosahedron $[5/2|2,3]$ skel $\cong U_{34}, U_{35}$

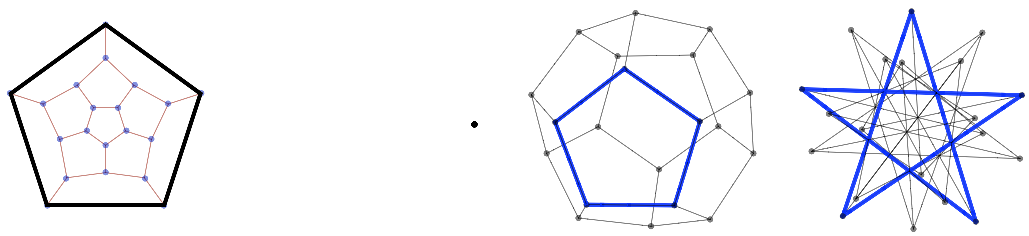
λ	Dim	Min Poly	#V: 12	V Fig: 3^5	Description	$(r_0, r_1, \dots)$
5	1	$x-5$	1	$3^5/0$	dot	$\frac{1}{12}(1,1,1,1)$
$\sqrt{5}$	3	x^2-5	12	3^5	U_{22} (covert)	$\frac{1}{20}(5,\sqrt{5},-\sqrt{5},-5)$
$-\sqrt{5}$	3	x^2-5	12	3^5	U_{53} (covert)	$\frac{1}{20}(5,-\sqrt{5},\sqrt{5},-5)$
-1	5	$x+1$	6	3^5	coincident with 5-simplex	$\frac{1}{12}(5,-1,-1,5)$



(a) Skeleton (b) $\lambda = 5$ (c) $\lambda = \sqrt{5} : U_{22}$ (d) $\lambda = -\sqrt{5} : U_{53}$

FIGURE 21. The U_{22}/U_{53} skeleton (a) and its low-dimensional spectral realizations.

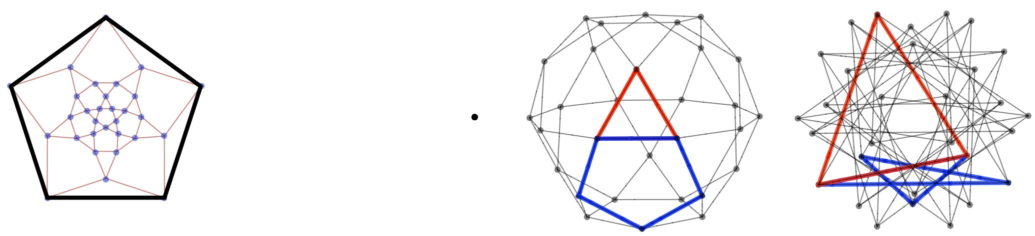
		U_{23}	Dodecahedron			$[3\|2, 5]$
		U_{52}	Great Stellated Dodecahedron			$[3\|2, 5/2]$
λ	Dim	Min Poly	#V: 20	V Fig: 5^3	Description	$(r_0, r_1, \dots)$
3	1	$x-3$	1	$5^3/0$	dot	$\frac{1}{20}\,(1, 1, 1, 1, 1, 1)$
$\sqrt{5}$	3	x^2-5	20	5^3	U_{23} (covert)	$\frac{1}{20}\,(3, \sqrt{5}, 1, -1, -\sqrt{5}, -3)$
$-\sqrt{5}$	3	x^2-5	20	$(5/2)^3$	U_{52} (covert)	$\frac{1}{20}\,(3, -\sqrt{5}, 1, -1, \sqrt{5}, -3)$
0	4	x	5	-	-	$\frac{1}{10}\,(2, 0, -1, 1, 0, -2)$
-2	4	$x+2$	10	-	Petersen	$\frac{1}{30}\,(6, -4, 1, 1, -4, 6)$
1	5	$x-1$	10	-	Petersen	$\frac{1}{20}\,(3, 1, -1, -1, 1, 3)$



(a) Skeleton (b) $\lambda = 3$ (c) $\lambda = \sqrt{5} : U_{23}$ (d) $\lambda = -\sqrt{5} : U_{52}$

FIGURE 22. The U_{23} / U_{52} skeleton (a) and its low-dimensional spectral realizations.

U_{24} Icosidodecahedron			$[2\|3,5]$		
U_{54} Great Icosidodecahedron			$[2\|5/2,3]$	skel $\cong U_{49}, U_{71}, U_{51}, U_{70}$	
λ	Dim	Min Poly	#V: 30	V Fig: $(3.5)^2$	Description
4	1	$x-4$	1	$(3.5)^2/0$	dot
$1+\sqrt{5}\approx 3.236$ $1-\sqrt{5}\approx -1.236$	3	x^2-2x-4	30	$(3.5)^2$ $(3.5/2)^2$	U_{24} (co I) U_{54} (co I)
-1	4	$x+1$	5	-	- (sup I)
1	4	$x-1$	30	-	- (sub I)
2	5	$x-2$	15	-	-
-2	10	$x+2$	30	-	-

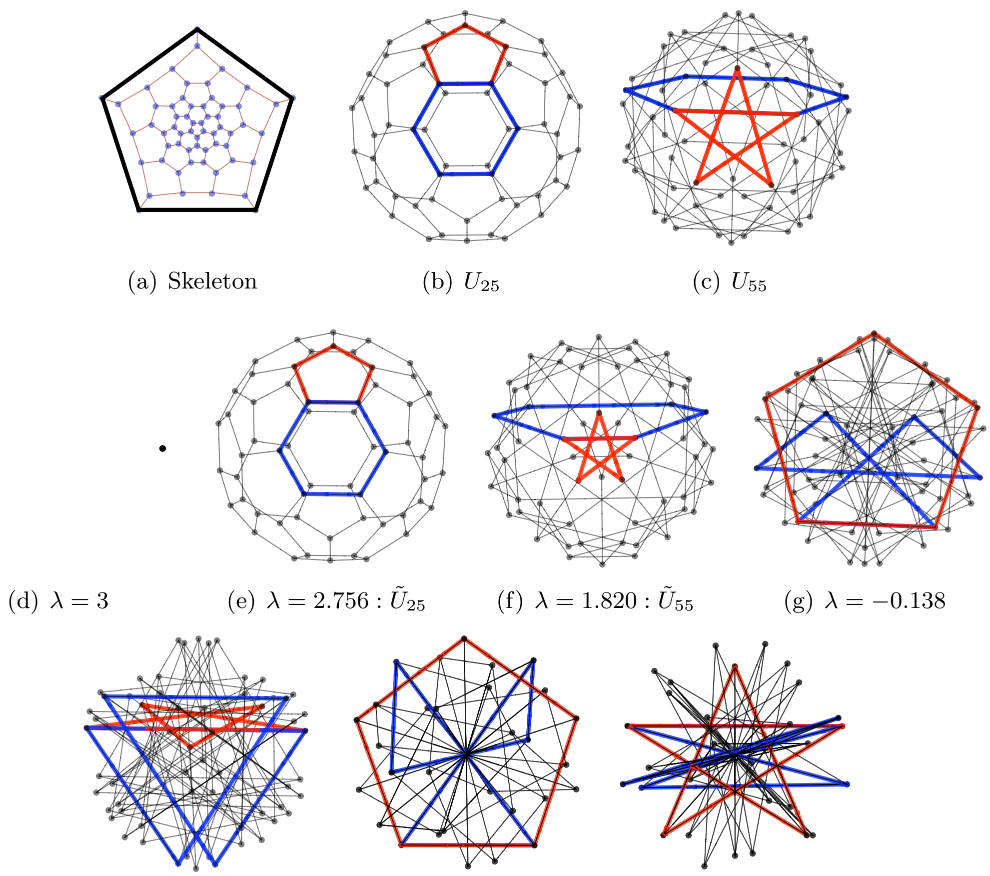


(a) Skeleton (b) $\lambda = 4$ (c) $\lambda = 1+\sqrt{5} : U_{24}$ (d) $\lambda = 1-\sqrt{5} : U_{54}$

FIGURE 23. The U_{24} / U_{54} skeleton (a) and its low-dimensional spectral realizations. (The $(1+\sqrt{5})$- and $(1-\sqrt{5})$-realizations are classical.)

U_{25}	Truncated Icosahedron	$[2, 5\|3]$
U_{55}	Great Truncated Icosahedron	$[2, 5/2\|3]$

λ	Dim	Min Poly	#V: 60	V Fig: 5.6^2	Description
3	1	$x-3$	1	$(5.6^2)/0$	dot
2.756 1.820 -0.138 -1.438	3	$x^4-3x^3-2x^2+7x+1$	60	$5.\tilde{6}^2$ $5/2.\tilde{6}^2$ - -	$\tilde{U}_{25}$ $\tilde{U}_{55}$ - -
$\frac{1}{2}\left(-3+\sqrt{5}\right) \approx -0.381$ $\frac{1}{2}\left(-3-\sqrt{5}\right) \approx -2.618$	3	x^2+3x+1	30	- -	- (co I) - (co I)
$\frac{1}{2}\left(-1+\sqrt{17}\right) \approx 1.561$ $\frac{1}{2}\left(-1-\sqrt{17}\right) \approx -2.561$	4	x^2+x-4	60	- -	- (co II) - (co II)
-2	4	$x+2$	10	-	-
$\frac{1}{2}\left(-1+\sqrt{5}\right) \approx 0.618$ $\frac{1}{2}\left(-1-\sqrt{5}\right) \approx -1.618$	5	x^2+x-1	60	- -	- (co III) - (co III)
$\frac{1}{2}\left(1+\sqrt{13}\right) \approx 2.302$ $\frac{1}{2}\left(1-\sqrt{13}\right) \approx -1.302$	5	x^2-x-3	30	- -	- -
1	9	$x-1$	30	-	-



(a) Skeleton (b) U_{25} (c) U_{55}

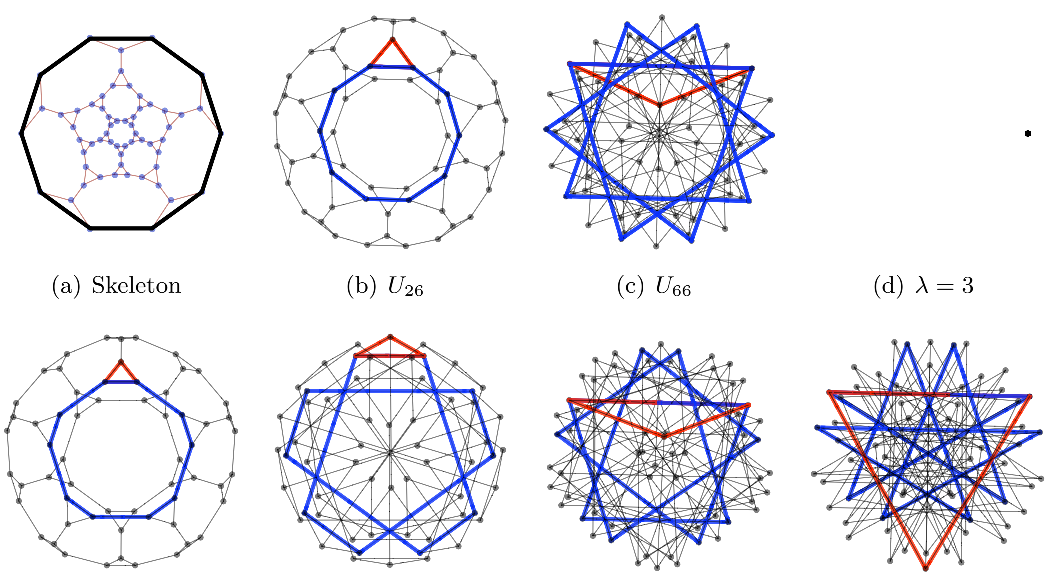
(d) $\lambda = 3$ (e) $\lambda = 2.756 : \tilde{U}_{25}$ (f) $\lambda = 1.820 : \tilde{U}_{55}$ (g) $\lambda = -0.138$

(h) $\lambda = -1.438$ (i) $\lambda = \frac{1}{2}(-3 + \sqrt{5})$ (j) $\lambda = \frac{1}{2}(-3 - \sqrt{5})$

FIGURE 24. The U_{25} / U_{55} skeleton (a), its classical realizations (b, c), and its low-dimensional spectral realizations. (The 2.756- and 1.820-realizations are only pseudo-classical: their hexagonal faces are not regular.)

U_{26} Truncated Dodecahedron $[2, 3|5]$
U_{66} Great Stellated Truncated Dodecahedron $[2, 3|5/3]$

λ	Dim	Min Poly	#V: 60	V Fig: 3.10^2	Description
3	1	$x-3$	1	$(3.10^2)/0$	dot
2.842 1.506 -0.506 -1.842	3	$x^4-2x^3-5x^2+6x+4$	60	$3.\tilde{10}^2$ - - -	$\tilde{U}_{26}$ - - -
$\frac{1}{2}\left(1+\sqrt{13}\right)\approx 2.302$ $\frac{1}{2}\left(1-\sqrt{13}\right)\approx -1.302$	4	x^2-x-3	60	- -	- (co I) - (co I)
$\frac{1}{2}\left(1+\sqrt{5}\right)\approx 1.618$ $\frac{1}{2}\left(1-\sqrt{5}\right)\approx -0.618$	4	x^2-x-1	30	- -	- (co II) - (co II)
$\frac{1}{2}\left(1+\sqrt{17}\right)\approx 2.561$ $\frac{1}{2}\left(1-\sqrt{17}\right)\approx -1.561)$	5	x^2-x-4	30	- -	- -
0	10	x	30	-	-
-2	11	$x+2$	60	-	-



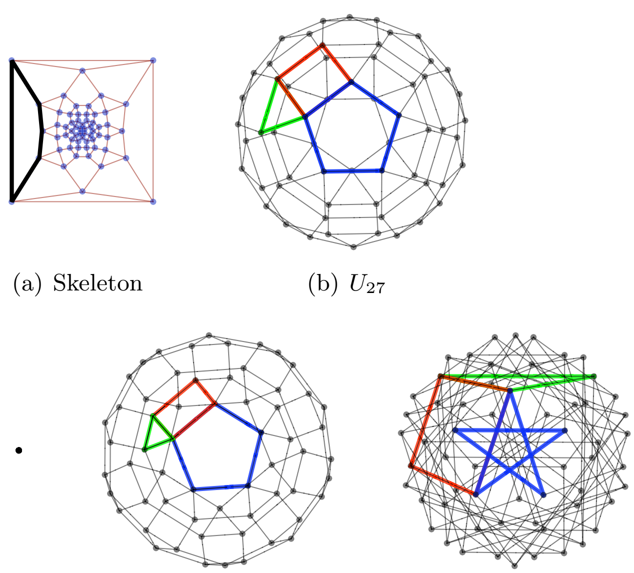
(a) Skeleton (b) U_{26} (c) U_{66} (d) $\lambda = 3$

(e) $\lambda = 2.842$ (f) $\lambda = 1.506$ (g) $\lambda = -0.506$ (h) $\lambda = -1.842$

FIGURE 25. The U_{26} / U_{66} skeleton (a), its classical realizations (b, c), and its low-dimensional spectral realizations. (The 2.842-realization is only pseudo-classical: the decagonal faces are not regular.)

U_{27} Small Rhombicosidodecahedron $[3,5|2]$ skel $\cong U_{39}, U_{33}$

λ	Dim	Min Poly	#V: 60	V Fig: 3.4.5.4	Description
4	1	$x-4$	1	(3.4.5.4)/0	dot
$\frac{1}{2}\left(5+\sqrt{5}\right) \approx 3.618$ $\frac{1}{2}\left(5-\sqrt{5}\right) \approx 1.381$	3	x^2-5x+5	60	$3.\tilde{4}.5.\tilde{4}$ -	$\tilde{U}_{27}$ (co I) - (co I)
$\sqrt{5} \approx 2.236$ $-\sqrt{5} \approx -2.236$	4	x^2-5	60	- -	- (co II) - (co II)
1	4	$x-1$	30	-	-
-1	4	$x+1$	5	-	-
2.925 0.551 -2.477	5	x^3-x^2-7x+4	30	- - -	- - -
0	6	x	12	-	-
$\frac{1}{2}\left(-3+\sqrt{5}\right) \approx -0.381$ $\frac{1}{2}\left(-3-\sqrt{5}\right) \approx -2.618$	8	x^2+3x+1	60	- -	- (co III) - (co III)



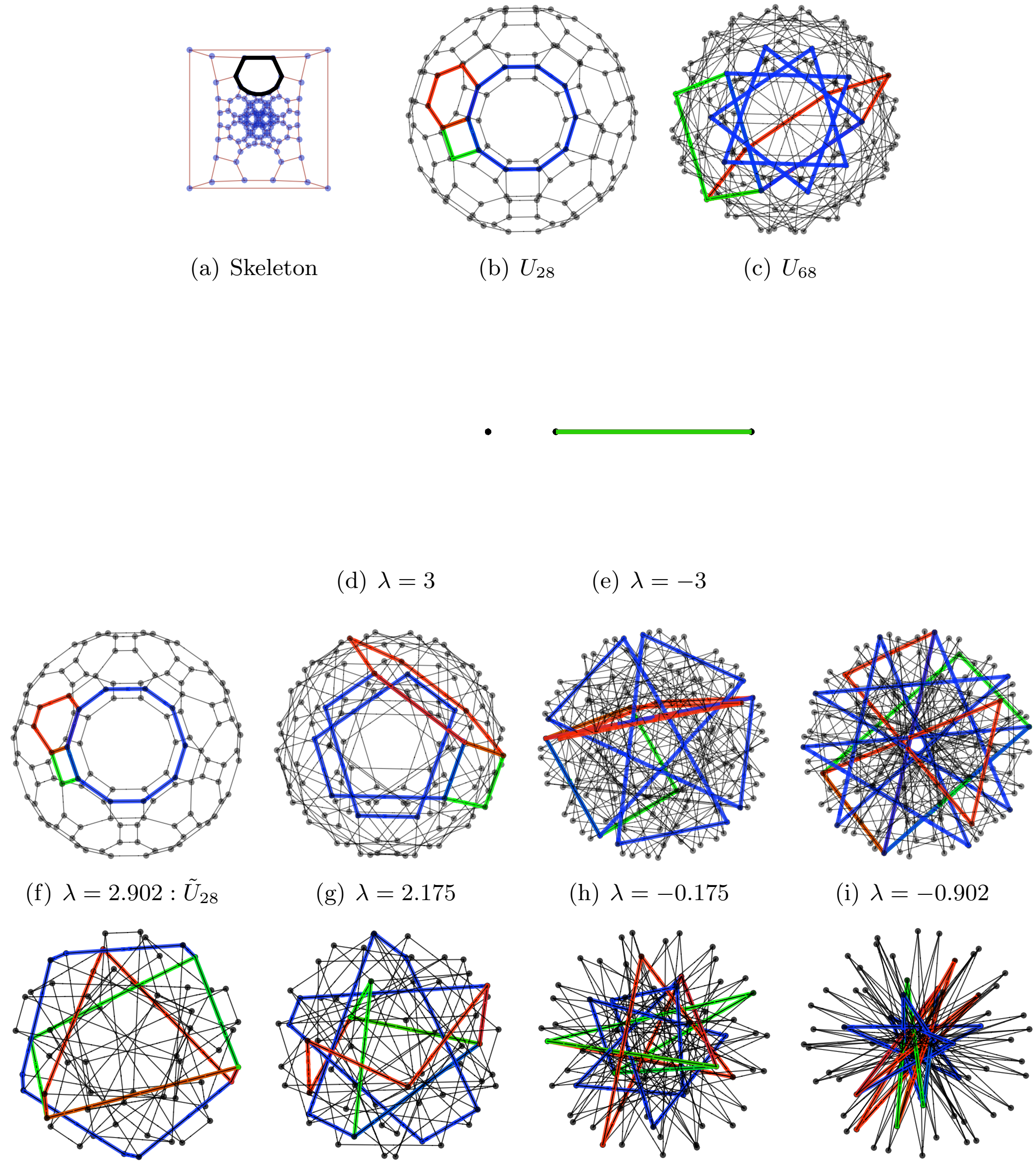
(a) Skeleton (b) U_{27}

(c) $\lambda = 4$ (d) $\lambda = \frac{1}{2}\left(5+\sqrt{5}\right) : \tilde{U}_{27}$ (e) $\lambda = \frac{1}{2}\left(5-\sqrt{5}\right)$

FIGURE 26. The U_{27} skeleton (a) and its low-dimensional spectral realizations. (The $\frac{1}{2}(5+\sqrt{5})$-realization is only pseudo-classical: the triangular faces have non-square neighbors.)

U_{28} Great Rhombicosidodecahedron $[2,3,5|]$
U_{68} Great Truncated Icosidodecahedron $[5/3,2,3|]$

λ	Dim	Min Poly	#V: 120	V Fig: 4.6.10	Description
3	1	$x-3$	1	(4.6.10)/0	dot
-3	1	$x+3$	2	-	stick
2.902 2.175 -0.175 -0.902	3	$x^4-4x^3+x^2+6x+1$	120	$\tilde{4}.\tilde{6}.\tilde{10}$ - - -	$\tilde{U}_{28}$ (co I, sup I) - (co I) - (co I) - (co I)
0.902 0.175 -2.175 -2.902	3	$x^4+4x^3+x^2-6x+1$	60	- - - -	- (co II, sub I) - (co II) - (co II) - (co II)
2.545 0.439 -0.830 -2.154	4	x^4-6x^2-2x+2	120	- - - -	- (sup III) - (sup X) - (sup VII) - (sup IV)
2.154 0.830 -0.439 -2.545	4	x^4-6x^2+2x+2	60	- - - -	- (sub IV) - (sub VII) - (sub X) - (sub III)
1.828 0.466 -0.684 -1.888 -2.721	5	$x^5+3x^4-3x^3-11x^2-x+3$	120	- - - - -	- (sup VI) - (sup IX) - (sup VIII) -(sup V) - (sup II)
2.721 1.888 0.684 -0.466 -1.828	5	$x^5-3x^4-3x^3+11x^2-x-3$	60	- - - - -	- (sub II) - (sub V) - (sub VIII) - (sub IX) - (sub VI)
1	6	$x-1$	12	-	- (co III)
-1	6	$x+1$	12	-	- (co III)



(a) Skeleton (b) U_{28} (c) U_{68}

(d) $\lambda = 3$ (e) $\lambda = -3$

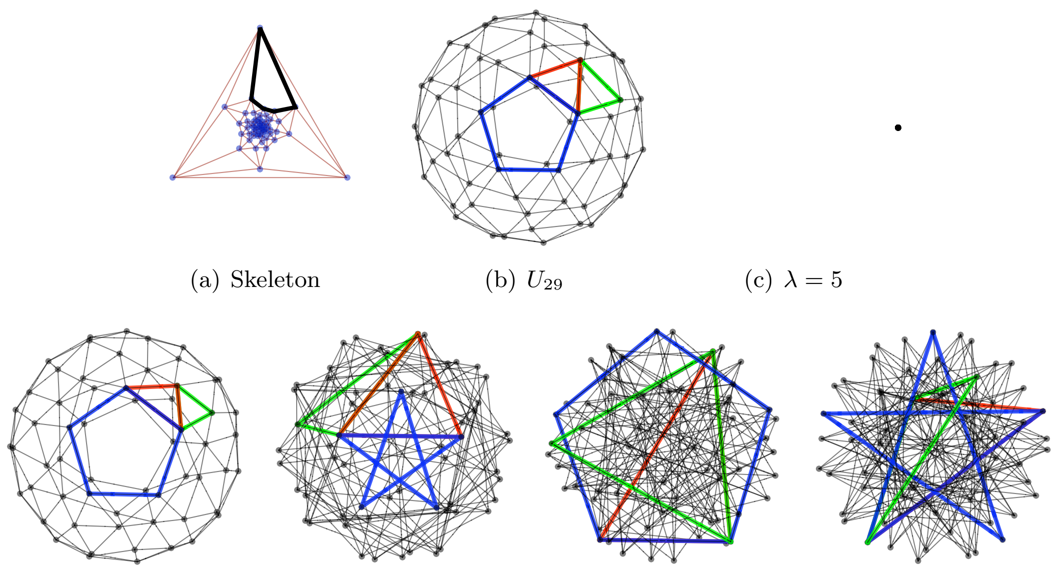
(f) $\lambda = 2.902 : \tilde{U}_{28}$ (g) $\lambda = 2.175$ (h) $\lambda = -0.175$ (i) $\lambda = -0.902$

(j) $\lambda = 0.902$ (k) $\lambda = 0.175$ (l) $\lambda = -2.175$ (m) $\lambda = -2.902$

FIGURE 27. The U_{28} / U_{68} skeleton (a), its classical realizations (b, c) and its low-dimensional spectral realizations. (The 2.902-realization only pseudo-classical: none of its faces are regular.)

U_{29} Snub Dodecahedron $[|2, 3, 5]$

λ	Dim	Min Poly	#V: 60	V Fig: 5.3^4	Description
5	1	$x - 5$	1	$(5.3^4)/0$	dot
4.487	3	$x^4 - 2x^3 - 13x^2 + 4x + 19$	60	$5.\tilde{3}.3.\tilde{3}.\tilde{3}$	$\tilde{U}_{29}$
1.322				-	-
-1.251				-	-
-2.558				-	-
2.716	4	$x^4 - 8x^2 - 2x + 10$	60	-	-
1.070				-	-
-1.507				-	-
-2.280				-	-
3.576	5	$x^5 + x^4 - 11x^3 - 19x^2 - x + 1$	60	-	-
0.195				-	-
-0.285				-	-
-2.135				-	-
-2.351				-	-
-1	6	$x + 1$	12	-	-

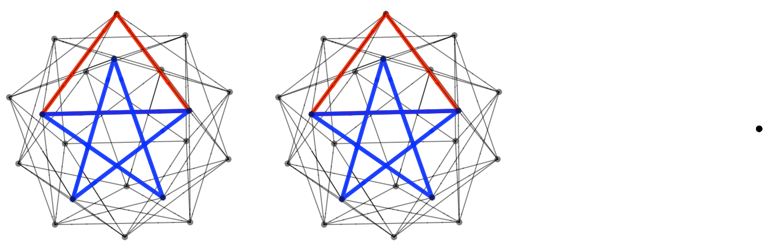


(a) Skeleton (b) U_{29} (c) $\lambda = 5$

(d) $\lambda = 4.487 : \tilde{U}_{29}$ (e) $\lambda = 1.322$ (f) $\lambda = -1.251$ (g) $\lambda = -2.558$

FIGURE 28. The U_{29} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations. (The 4.487-realization is only pseudo-classical: the triangular neighbors of the pentagonal faces are not regular.)

U_{30}	Small Ditrigonal Icosidodecahedron	$[3\|5/2, 3]$	skel $\cong U_{41}, U_{47}$		
λ	Dim	Min Poly	#V: 20	V Fig: $(3.5)^3$	Description
6	1	$x-6$	1	$(3.5)^3/0$	dot
1	4	$x-1$	-	-	-
-3	4	$x+3$	-	-	-
-2	5	$x+2$	-	-	-
2	6	$x-2$	20	-	$\hat{U}_{30}$

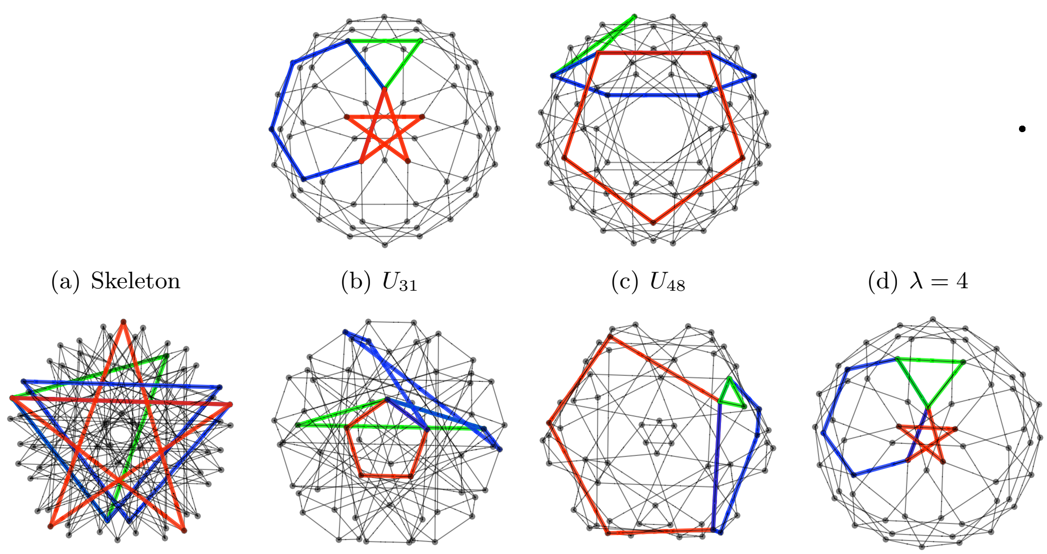


(a) Skeleton (b) $U_{30}\ |\lambda = 2$ (c) $\lambda = 6$

FIGURE 29. The U_{30} skeleton (a), its classical realization (b), and its low-dimensional spectral realization. The classical realization is eigenic (and harmonious), but does not represent a spectral realization.

U_{31} (6,5,6,3) Small Icosicosidodecahedron
U_{48} (5,6,3,6) Great Icosicosidodecahedron $[3/2, 5|3]$ skel $\cong U_{42}, U_{43}, U_{50}, U_{63}$

λ	Dim	Min Poly	#V: 60	V Fig: 6.5.6.3	Desc
4	1	$x-4$	1	$(6.5.6.3)\,/0$	dot
-1.945	3	$x^4-5x^3+x^2+20x-16$	-	-	-
0.914	3		-	-	-
2.703	3		60	$\tilde{6}.5.\tilde{6}.3/2$	$\tilde{U}_{48}$
3.327	3		60	$\tilde{6}.5/2.\tilde{6}.3$	$\tilde{U}_{31}$
$-\sqrt{6}\approx-2.449$	4	x^2-6	-	-	-
$\sqrt{6}\approx 2.449$			-	-	-
$-\sqrt{2}\approx-1.414$	4	x^2-2	-	-	-
$\sqrt{2}\approx 1.414$			-	-	-
$\frac{1}{2}\left(1-\sqrt{13}\right)\approx-1.302$	5	x^2-x-3	-	-	-
$\frac{1}{2}\left(1+\sqrt{13}\right)\approx 2.302$			-	-	-
0	5	x	-	-	-
$\frac{1}{2}\left(-3-\sqrt{5}\right)\approx-2.618$	8	x^2+3x+1	-	-	-
$\frac{1}{2}\left(-3+\sqrt{5}\right)\approx-0.381$			-	-	-



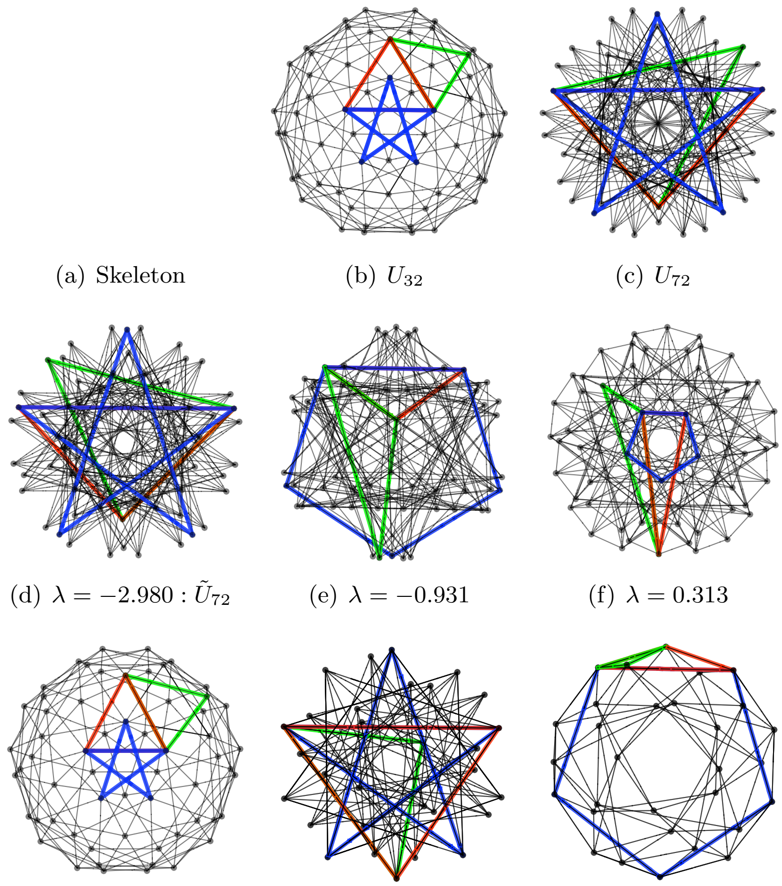
(a) Skeleton (b) U_{31} (c) U_{48} (d) $\lambda=4$

(e) $\lambda=-1.945$ (f) $\lambda=0.914$ (g) $\lambda=2.703:\tilde{U}_{48}$ (h) $\lambda=3.327:\tilde{U}_{31}$

FIGURE 30. The U_{31} / U_{48} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations. (The 2.703- and 3.327-realizations are only pseudo-classical: their hexagonal faces are non-regular.)

U_{32} Small Snub Icosicosidodecahedron $[|5/2, 3, 3]$
U_{72} Small Retrosnub Icosicosidodecahedron $[|3/2, 3/2, 5/2]$

λ	Dim	Min Poly	#V: 60	V Fig: 5.3^5	Desc
6	1	$x-6$	1	$(5.3^5)/0$	dot
-2.980 -0.931 0.313 4.598	3	$x^4-x^3-15x^2-8x+4$	60 - - 60	$(5/3,\tilde{3},3,\tilde{3},3,\tilde{3})/2$ - - $5/2,\tilde{3},3,\tilde{3},3,\tilde{3}$	$\tilde{U}_{72}$ - - $\tilde{U}_{32}$
$\frac{1}{2}\left(1-3\sqrt{5}\right)\approx-2.854$ $\frac{1}{2}\left(1+3\sqrt{5}\right)\approx 3.854$	3	x^2-x-11	- -	- -	- -
$\frac{1}{2}\left(3-\sqrt{17}\right)\approx-0.561$ $\frac{1}{2}\left(3+\sqrt{17}\right)\approx 3.561$	4	x^2-3x-2	- -	- -	- -
1	4	$x-1$	-	-	-
$\frac{1}{2}\left(-3-\sqrt{5}\right)\approx-2.618$ $\frac{1}{2}\left(-3+\sqrt{5}\right)\approx-0.381$	5	x^2+3x+1	- -	- -	- -
$\frac{1}{2}\left(1-\sqrt{13}\right)\approx-1.302$ $\frac{1}{2}\left(1+\sqrt{13}\right)\approx 2.302$	5	x^2-x-3	- -	- -	- -
-2	9	$x+2$	-	-	-



(a) Skeleton (b) U_{32} (c) U_{72}

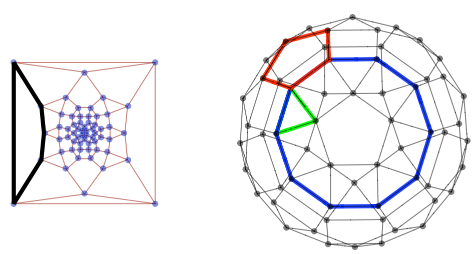
(d) $\lambda = -2.980 : \tilde{U}_{72}$ (e) $\lambda = -0.931$ (f) $\lambda = 0.313$

(g) $\lambda = 4.598 : \tilde{U}_{32}$ (h) $\lambda = \frac{1}{2}\left(1 - 3\sqrt{5}\right)$ (i) $\lambda = \frac{1}{2}\left(1 + \sqrt{5}\right)$

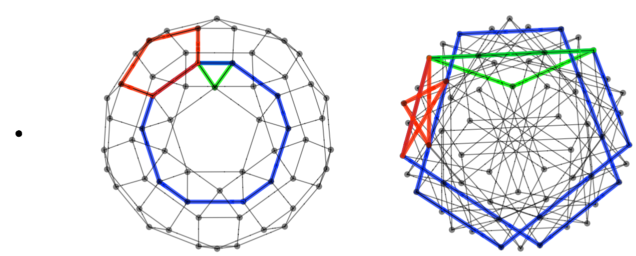
Figure 31. The U_{32} / U_{72} skeleton (a), its classical realizations (b,c), and its low-dimensional spectral realization. (The 4.598- and -2.980-realizations are only pseudo-classical: the triangular neighbors of their pentagrammic faces are non-regular.)

U_{33} Small Dodecicosidodecahedron $[3/2, 5|5]$ skel $\cong U_{27}, U_{39}$

λ	Dim	Min Poly	#V: 60	V Fig: 5.10.3.10	Desc
4	1	$x-4$	1	(5.10.3.10)/0	dot
$\frac{1}{2}\left(5+\sqrt{5}\right) \approx 3.618$ $\frac{1}{2}\left(5-\sqrt{5}\right) \approx 1.381$	3	x^2-5x+5	60	$5.\tilde{10}.3/2.\tilde{10}$ -	$\tilde{U}_{33}$ (co I) - (co I)
$\sqrt{5} \approx 2.236$ $-\sqrt{5} \approx -2.236$	4	x^2-5	60	- -	- (co II) - (co II)
1	4	$x-1$	30	-	-
-1	4	$x+1$	5	-	-
2.925 0.551 -2.477	5	x^3-x^2-7x+4	30	- - -	- - -
0	6	x	12	-	-
$\frac{1}{2}\left(-3+\sqrt{5}\right) \approx -0.381$ $\frac{1}{2}\left(-3-\sqrt{5}\right) \approx -2.618$	8	x^2+3x+1	60	- -	- (co III) - (co III)



(a) Skeleton (b) U_{33}

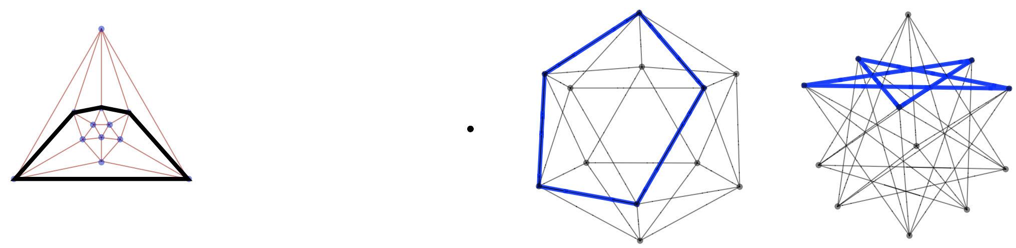


(c) $\lambda = 4$ (d) $\lambda = \frac{1}{2}\left(5+\sqrt{5}\right) : \tilde{U}_{33}$ (e) $\lambda = \frac{1}{2}\left(5-\sqrt{5}\right)$

FIGURE 32. The U_{33} skeleton (a) and its low-dimensional spectral realizations. (The $\frac{1}{2}(5+\sqrt{5})$-realization is only pseudo-classical: the decagonal faces are non-regular.)

U_{34} Small Stellated Dodecahedron $[5|2, 5/2]$
U_{35} Great Dodecahedron $[5/2|2, 5]$ skel $\cong U_{22}, U_{53}$

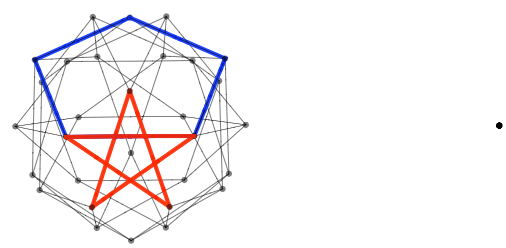
λ	Dim	Min Poly	#V: 12	V Fig: 5^5	Description	$(r_0, r_1, \dots)$
5	1	$x - 5$	1	$5^5/0$	dot	$\frac{1}{12}\,(1, 1, 1, 1)$
$\sqrt{5}$	3	$x^2 - 5$	12	$(5/2)^5$	U_{35} (covert)	$\frac{1}{20}\,(5, \sqrt{5}, -\sqrt{5}, -5)$
$-\sqrt{5}$	3	$x^2 - 5$	12	$(5/2)^5$	U_{34} (covert)	$\frac{1}{20}\,(5, -\sqrt{5}, \sqrt{5}, -5)$
-1	5	$x + 1$	6	-	coincident w/ 5-simplex	$\frac{1}{12}\,(5, -1, -1, 5)$



(a) Skeleton (b) $\lambda = 5$ (c) $\lambda = \sqrt{5} : U_{35}$ (d) $\lambda = -\sqrt{5} : U_{34}$

FIGURE 33. The U_{34} / U_{35} skeleton (a) and its low-dimensional spectral realizations.

U_{36} Dodecadodecahedron $[2\|5/2, 5]$ skel $\cong U_{62}$, U_{65}					
λ	Dim	Min Poly	#V: 30	V Fig: 5^4	Description
4	1	$x-4$	1	$5^4/0$	dot
-3	4	$x+3$	-	-	-
-1	4	$x+1$	-	-	-
-2	5	$x+2$	-	-	-
0	5	x	-	-	-
2	11	$x-11$	30	-	$\hat{U}_{36}$

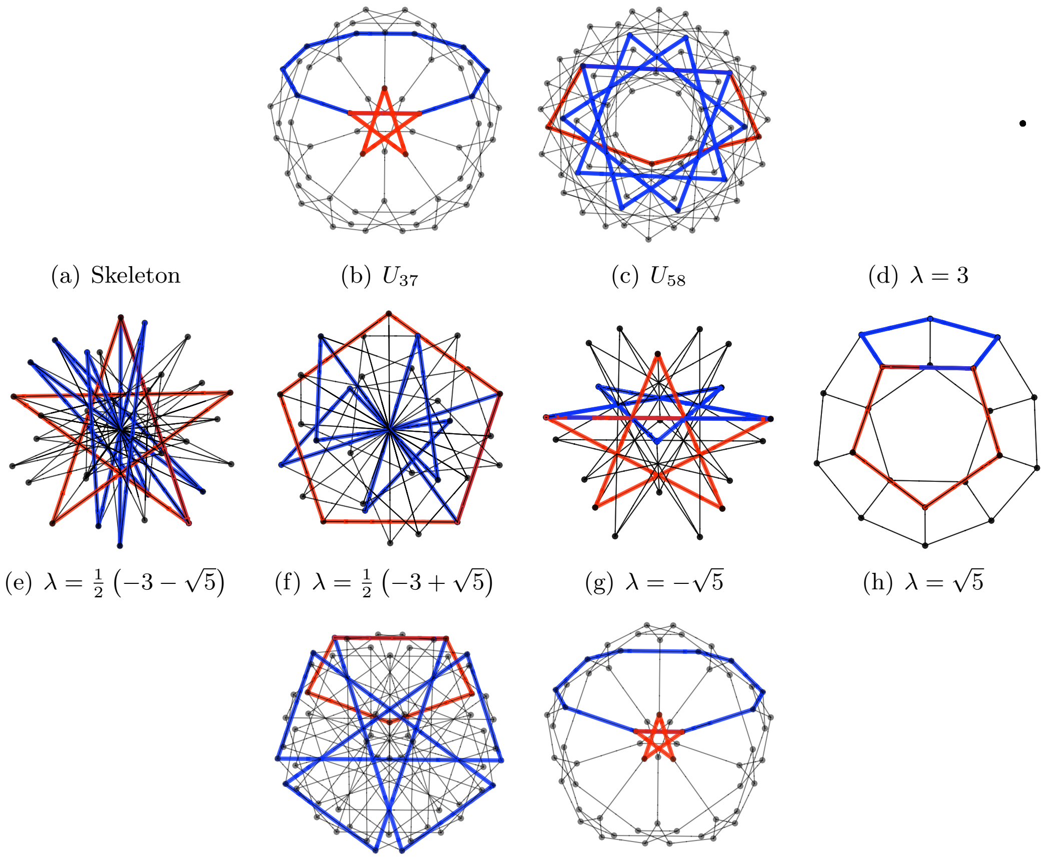


(a) Skeleton (b) $U_{36} : \lambda = 2$ (c) $\lambda = 4$

FIGURE 34. The U_{36} skeleton (a), its classical realization (b), and its low-dimensional spectral realization. The classical realization is eigenic (and harmonious), but does not represent a spectral realization. (See note with Figure 52 regarding U_{62} and U_{65} and this phenomenon.)

U_{37} Truncated Great Dodecahedron [25/2|5]
U_{58} Small Stellated Truncated Dodecahedron [25|5/3]

λ	Dim	Min Poly	#V: 60	V Fig: 5.10^2	Description
3	1	$x-3$	1	(5.10.10)/0	dot
$\frac{1}{2}\left(-3-\sqrt{5}\right) \approx -2.618$ $\frac{1}{2}\left(-3+\sqrt{5}\right) \approx -0.381$	3	x^2+3x+1	- -	- -	- -
$-\sqrt{5} \approx -2.236$ $\sqrt{5} \approx 2.236$	3	x^2-5	- -	- -	- -
$\frac{1}{2}\left(3-\sqrt{5}\right) \approx 0.381$ $\frac{1}{2}\left(3+\sqrt{5}\right) \approx 2.618$	3	x^2-3x+1	- 60	- $5.\tilde{10}.\tilde{10}$	- $\tilde{U}_{37}$
-2	4	$x+2$	-	-	-
-1	4	$x+1$	-	-	-
0	4	x	-	-	-
$\frac{1}{2}\left(-1-\sqrt{13}\right) \approx -2.302$ $\frac{1}{2}\left(-1+\sqrt{13}\right) \approx 1.302$	5	x^2+x-3	- -	- -	- -
$\frac{1}{2}\left(1-\sqrt{13}\right) \approx -1.302$ $\frac{1}{2}\left(1+\sqrt{13}\right) \approx 2.302$	5	x^2-x-3	- -	- -	- -
1	9	$x-1$	-	-	-



(a) Skeleton (b) U_{37} (c) U_{58} (d) $\lambda = 3$

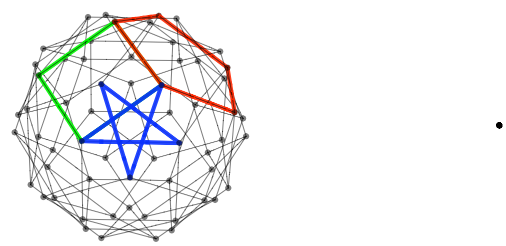
(e) $\lambda = \frac{1}{2}\left(-3-\sqrt{5}\right)$ (f) $\lambda = \frac{1}{2}\left(-3+\sqrt{5}\right)$ (g) $\lambda = -\sqrt{5}$ (h) $\lambda = \sqrt{5}$

(i) $\lambda = \frac{1}{2}\left(3-\sqrt{5}\right)$ (j) $\lambda = \frac{1}{2}\left(3+\sqrt{5}\right) : \tilde{U}_{37}$

FIGURE 35. The U_{37} / U_{58} skeleton (a), its classical realizations (b, c), and its low-dimensional spectral realization. (The $\frac{1}{2}\left(3+\sqrt{5}\right)$-realization is only pseudo-classical: the decagonal faces are non-regular.)

U_{38} Rhombidodecadodecahedron $[5/2, 5|2]$ skel $\cong U_{44}, U_{56}$

λ	Dim	Min Poly	#V: 60	V Fig: $(4.5)^2$	Description
4	1	$x-4$	1	$(4.5)^2/0$	dot
$-1-\sqrt{5} \approx -3.236$ $-1+\sqrt{5} \approx 1.236$	4	x^2+2x-4	- -	- -	- -
-2	4	$x+2$	-	-	-
-3	5	$x+3$	-	-	-
$\frac{1}{2}\left(1-\sqrt{17}\right) \approx -1.561$ $\frac{1}{2}\left(1+\sqrt{17}\right) \approx 2.561$	5	x^2-x-4	- -	- -	- -
-1	6	$x+1$	-	-	-
3	6	$x-3$	-	-	-
0	10	x	-	-	-
1	10	$x-10$	-	-	-

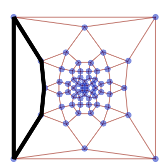


(a) Skeleton (b) U_{38} (c) $\lambda = 4$

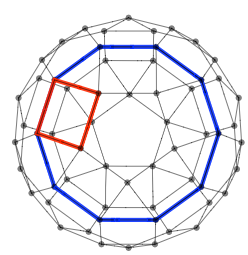
FIGURE 36. The U_{38} skeleton (a), its classical realization (b), and its low-dimensional spectral realization.

U_{39} Small Rhombidodecahedron $[2, 5/2, 5|]$ skel $\cong U_{27}, U_{33}$

λ	Dim	Min Poly	#V: 60	V Fig: $(4.10)^2$	Description
4	1	$x-4$	1	$(4.10)^2/0$	dot
$\frac{1}{2}\left(5+\sqrt{5}\right) \approx 3.618$ $\frac{1}{2}\left(5-\sqrt{5}\right) \approx 1.381$	3	x^2-5x+5	60	$\tilde{4}.\tilde{10}.\tilde{4}/3.\tilde{10}/9$ -	$\tilde{U}_{39}$ (co I) - (co I)
$\sqrt{5} \approx 2.236$ $-\sqrt{5} \approx -2.236$	4	x^2-5	60	- -	- (co II) - (co II)
1	4	$x-1$	30	-	-
-1	4	$x+1$	5	-	-
2.925 0.551 -2.477	5	x^3-x^2-7x+4	30	- - -	- - -
0	6	x	12	-	-
$\frac{1}{2}\left(-3+\sqrt{5}\right) \approx -0.381$ $\frac{1}{2}\left(-3-\sqrt{5}\right) \approx -2.618$	8	x^2+3x+1	60	- -	- (co III) - (co III)

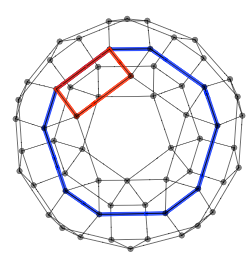


(a) Skeleton

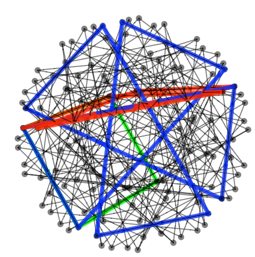


(b) U_{39}

(c) $\lambda = 4$



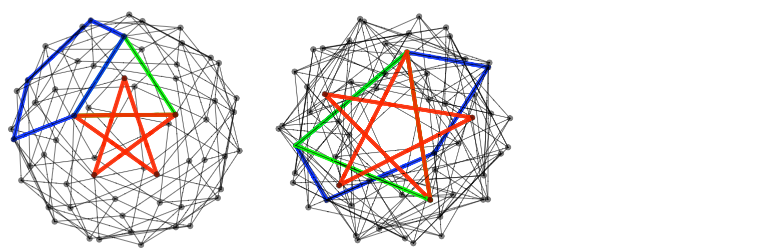
(d) $\lambda = \frac{1}{2}\left(5+\sqrt{5}\right) : \tilde{U}_{39}$



(e) $\lambda = \frac{1}{2}\left(5-\sqrt{5}\right)$

FIGURE 37. The U_{39} skeleton (a) and its low-dimensional spectral realizations. (The $\frac{1}{2}(5+\sqrt{5})$-realization is only pseudo-classical: none of the faces are regular.)

	U_{40} Snub Dodecadodecahedron	$[\|2, 5/2, 5]$			
	U_{60} Inverted Snub Dodecadodecahedron	$[\|5/3, 2, 5]$			
λ	Dim	Min Poly	#V: 60	V Fig: $3^2.5.3.5$	Description
5	1	$x - 5$	1	$(3^2.5.3.5)/0$	dot
-2.675 -1.539 1.214	4	$x^3 + 3x^2 - x - 5$	- - -	- - -	- - -
$-1-\sqrt{3} \approx -2.732$ $-1+\sqrt{3} \approx 0.732$	5	$x^2 + 2x - 2$	- -	- -	- -
-1.895 1.602 3.292	5	$x^3 - 3x^2 - 4x + 10$	- - -	- - -	- - -
$1-\sqrt{7} \approx -1.645$ $1+\sqrt{7} \approx 3.645$	6	$x^2 - 2x - 6$	- -	- -	- -
-1	10	$x + 1$	-	-	-

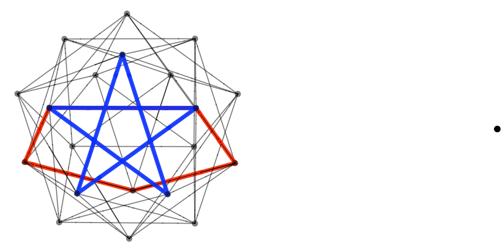


(a) Skeleton (b) U_{40} (c) U_{60} (d) $\lambda = 5$

FIGURE 38. The U_{40} / U_{60} skeleton (a), its classical realizations (b, c), and its low-dimensional spectral realization.

U_{41} Ditrigonal Dodecadodecahedron $[3|5/3, 5]$ skel $\cong U_{30}, U_{47}$

λ	Dim	Min Poly	#V: 20	V Fig: 5^6	Description
6	1	$x - 6$	1	$5^6/0$	dot
1	4	$x - 1$	-	-	-
-3	4	$x + 3$	-	-	-
-2	5	$x + 2$	-	-	-
2	6	$x - 2$	20	-	$\hat{U}_{41}$

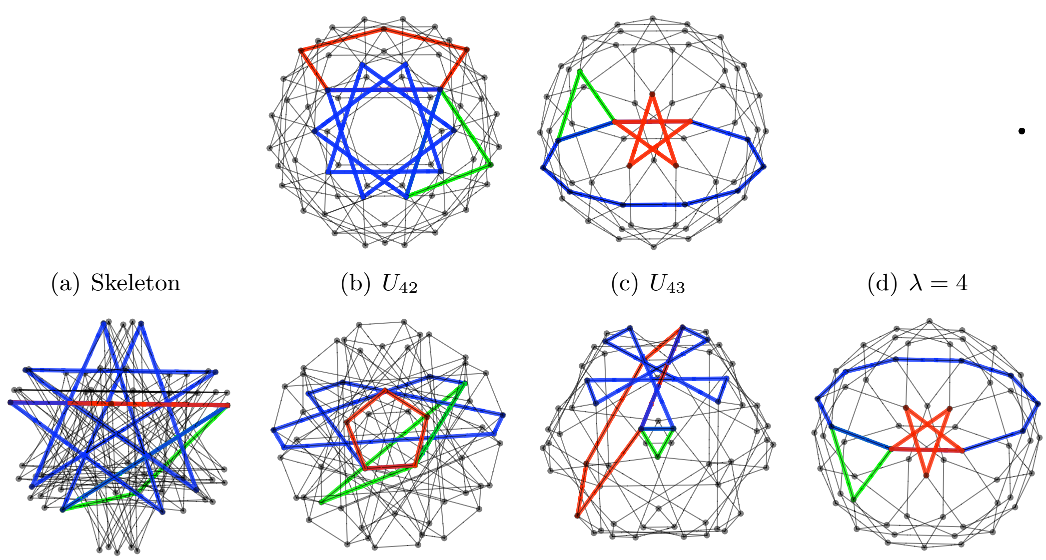


(a) Skeleton (b) U_{41} $|\lambda = 2$ (c) $\lambda = 6$

FIGURE 39. The U_{41} skeleton (a), its classical realization (b), and its low-dimensional spectral realization. The classical realization is eigenic (and harmonious), but does not represent a spectral realization.

U_{42} Great Ditrigonal Dodecicosidodecahedron $[3, 5|5/3]$
U_{43} Small Ditrigonal Dodecicosidodecahedron $[5/3, 3|5]$ skel $\cong U_{31}, U_{48}, U_{50}, U_{63}$

λ	Dim	Min Poly	#V: 60	V Fig: 3.10.5.10	Desc
4	1	$x-4$	1	$(3.10.5.10)/0$	dot
-1.945	3	$x^4-5x^3+x^2+20x-16$	-	-	-
0.914	3		-	-	-
2.703	3		-	-	-
3.327	3		60	$3.\widetilde{10}.5/3.\widetilde{10}$	$\tilde{U}_{43}$
$-\sqrt{6} \approx -2.449$	4	x^2-6	-	-	-
$\sqrt{6} \approx 2.449$			-	-	-
$-\sqrt{2} \approx -1.414$	4	x^2-2	-	-	-
$\sqrt{2} \approx 1.414$			-	-	-
$\frac{1}{2}\left(1-\sqrt{13}\right) \approx -1.302$	5	x^2-x-3	-	-	-
$\frac{1}{2}\left(1+\sqrt{13}\right) \approx 2.302$			-	-	-
0	5	x	-	-	-
$\frac{1}{2}\left(-3-\sqrt{5}\right) \approx -2.618$	8	x^2+3x+1	-	-	-
$\frac{1}{2}\left(-3+\sqrt{5}\right) \approx -0.381$			-	-	-

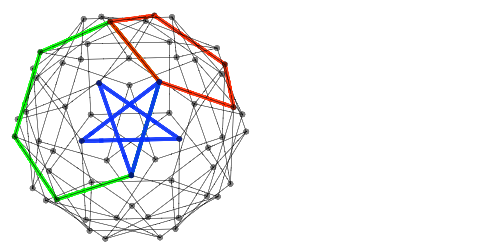


(a) Skeleton (b) U_{42} (c) U_{43} (d) $\lambda = 4$

(e) $\lambda = -1.945$ (f) $\lambda = 0.914$ (g) $\lambda = 2.703$ (h) $\lambda = 3.327 : \tilde{U}_{43}$

FIGURE 40. The U_{42} / U_{43} skeleton (a), its classical realizations (b, c), and its low-dimensional spectral realizations. (The 3.327-realization is only pseudo-classical: its decagonal faces are non-regular.)

U_{44} Icosidodecadodecahedron			$[5/3, 5\|3]$	skel $\cong U_{38}, U_{56}$	
λ	Dim	Min Poly	#V: 60	V Fig: $(5.6)^2$	Description
4	1	$x-4$	1	$(5.6)^2/0$	dot
$-1-\sqrt{5}\approx -3.236$ $-1+\sqrt{5}\approx 1.236$	4	x^2+2x-4	- -	- -	- -
-2	4	$x+2$	-	-	-
-3	5	$x+3$	-	-	-
$\frac{1}{2}\left(1-\sqrt{17}\right)\approx -1.561$ $\frac{1}{2}\left(1+\sqrt{17}\right)\approx 2.561$	5	x^2-x-4	- -	- -	- -
-1	6	$x+1$	-	-	-
3	6	$x-3$	-	-	-
0	10	x	-	-	-
1	10	$x-10$	-	-	-

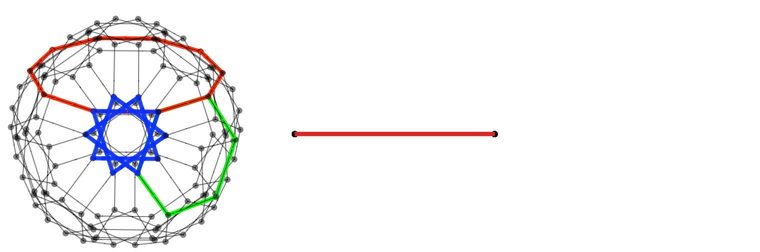


(a) Skeleton (b) U_{44} (c) $\lambda = 4$

FIGURE 41. The U_{44} skeleton (a), its classical realization (b), and its low-dimensional spectral realization.

U_{45} Icositruncated Dodecadodecahedron $[5/3, 3, 5|]$

λ	Dim	Min Poly	#V: 120	V Fig: 6.10^2	Description
-3	1	$x+3$	2	-	stick
3	1	$x-3$	1	$(6.10^2)/0$	dot
-2	4	$x+2$	-	-	-
2	4	$x-2$	-	-	-
-1	5	$x+1$	-	-	-
1	5	$x-1$	-	-	-
-2.675 -1.539 1.214	6	x^3+3x^2-x-5	- - -	- - -	- - -
-1.214 1.539 2.675	6	x^3-3x^2-x+5	- - -	- - -	- - -
0	8	x	-	-	-
$-1-\sqrt{2}\approx -2.414$ $-1+\sqrt{2}\approx 0.414$	9	x^2+2x-1	- -	- -	- -
$1-\sqrt{2}\approx -0.414$ $1+\sqrt{2}\approx 2.414$	9	x^2-2x-1	- -	- -	- -
$-\sqrt{3}$ $\sqrt{3}$	10	x^2-3	- -	- -	- -

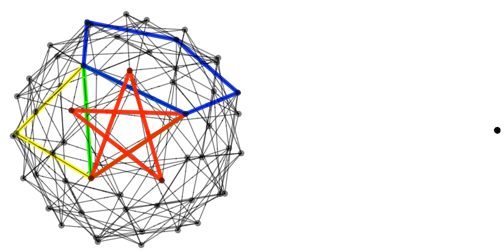


(a) Skeleton (b) U_{45} (c) $\lambda=-3$ (d) $\lambda=3$

FIGURE 42. The U_{45} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations.

U_{46} Snub Icosidodecadodecahedron $[|5/3, 3, 5]$

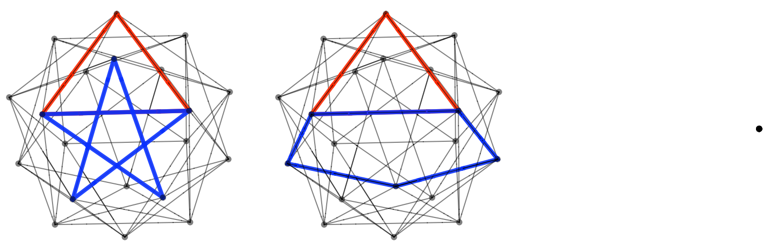
λ	Dim	Min Poly	#V: 60	V Fig: $3^3.5.3.5$	Description
6	1	$x-6$	1	$(3^3.5.3.5)/0$	dot
-3	4	$x+3$	-	-	-
1	4	$x-1$	-	-	-
-2	5	$x+2$	-	-	-
-1.525 -0.630 4.156	6	x^3-2x^2-8x-4	- - -	- - -	- - -
$-2\sqrt{2}\approx -2.828$ $2\sqrt{2}\approx 2.828$	9	x^2-8	- -	- -	- -
0	10	x	-	-	-



(a) Skeleton (b) U_{46} (c) $\lambda = 6$

FIGURE 43. The U_{46} skeleton (a), its classical realization (b), and its low-dimensional spectral realization.

U_{47}	Great Ditrigonal Icosidodecahedron		[3/2|3, 5]	skel $\cong U_{30}$, U_{41}	
λ	Dim	Min Poly	#V: 20	V Fig: $(3.5)^3$	Description
6	1	$x-6$	1	$(3.5)^3/0$	dot
1	4	$x-1$	-	-	-
-3	4	$x+3$	-	-	-
-2	5	$x+2$	-	-	-
2	6	$x-2$	20	-	$\hat{U}_{47}$



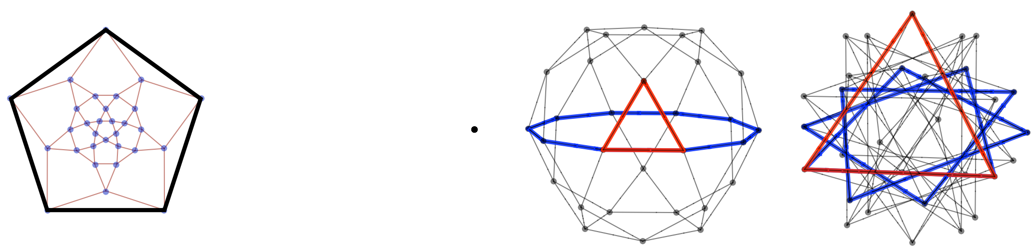
(a) Skeleton (b) U_{47} $|\lambda = 2$ (c) $\lambda = 6$

FIGURE 44. The U_{47} skeleton (a), its classical realization (b), and its low-dimensional spectral realization. The classical realization is eigenic (and harmonious), but does not represent a spectral realization.

U_{48} : see U_{31}

U_{49} Small Icosihemidodecahedron $[3/2, 3|5]$
U_{71} Great Icosihemidodecahedron $[3/2, 3|5/3]$ skel $\cong U_{24}, U_{54}, U_{51}, U_{70}$

λ	Dim	Min Poly	#V: 30	V Fig: $(3.10)^2$	Description
4	1	$x-4$	1	$(3.10)^2/0$	dot
$1+\sqrt{5} \approx 3.236$ $1-\sqrt{5} \approx -1.236$	3	x^2-2x-4	30	$(3.10.3/2.10)$ $(3.10/3)^2$	U_{49} (co I) U_{71} (co I)
-1	4	$x+1$	5	-	- (sup I)
1	4	$x-1$	30	-	- (sub I)
2	5	$x-2$	15	-	-
-2	10	$x+2$	30	-	-

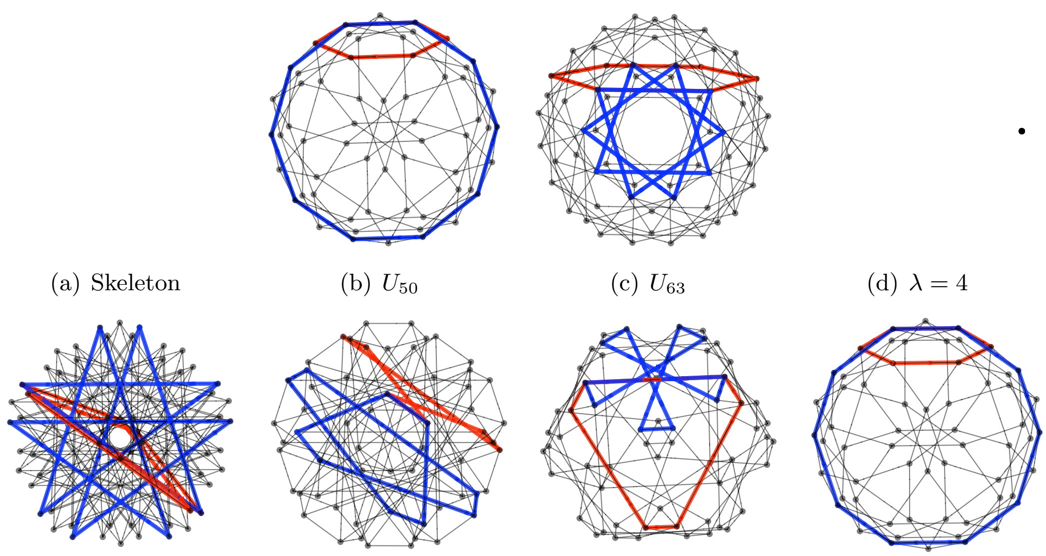


(a) Skeleton (b) $\lambda = 4$ (c) $\lambda = 1+\sqrt{5} : U_{49}$ (d) $\lambda = 1-\sqrt{5} : U_{71}$

FIGURE 45. The U_{49} / U_{71} skeleton (a) and its low-dimensional spectral realizations. (The $(1+\sqrt{5})$- and $(1-\sqrt{5})$-realizations are classical.)

U_{50} Small Dodecicosahedron $[3/2, 3, 5|]$
U_{63} Great Dodecicosahedron $[5/3, 5/2, 3|]$ skel $\cong U_{31}, U_{42}, U_{43}, U_{48}$

λ	Dim	Min Poly	#V: 60	V Fig: $(6.10)^2$	Desc
4	1	$x - 4$	1	$(6.10)^2/0$	dot
-1.945	3	$x^4 - 5x^3 + x^2 + 20x - 16$	-	-	-
0.914	3		-	-	-
2.703	3		-	-	-
3.327	3		60	$6.\tilde{10}.6/5.\tilde{10}/9$	$\tilde{U}_{50}$
$-\sqrt{6} \approx -2.449$	4	$x^2 - 6$	-	-	-
$\sqrt{6} \approx 2.449$			-	-	-
$-\sqrt{2} \approx -1.414$	4	$x^2 - 2$	-	-	-
$\sqrt{2} \approx 1.414$			-	-	-
$\frac{1}{2}\left(1 - \sqrt{13}\right) \approx -1.302$	5	$x^2 - x - 3$	-	-	-
$\frac{1}{2}\left(1 + \sqrt{13}\right) \approx 2.302$			-	-	-
0	5	x	-	-	-
$\frac{1}{2}\left(-3 - \sqrt{5}\right) \approx -2.618$	8	$x^2 + 3x + 1$	-	-	-
$\frac{1}{2}\left(-3 + \sqrt{5}\right) \approx -0.381$			-	-	-



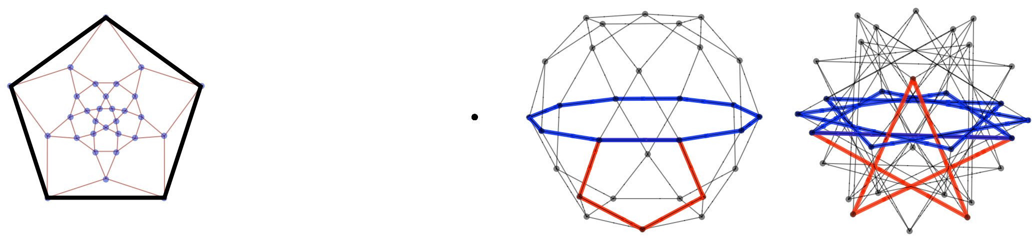
(a) Skeleton (b) U_{50} (c) U_{63} (d) $\lambda = 4$

(e) $\lambda = -1.945$ (f) $\lambda = 0.914$ (g) $\lambda = 2.703$ (h) $\lambda = 3.327 : \tilde{U}_{50}$

FIGURE 46. The U_{50} / U_{63} skeleton (a), its classical realizations (b, c), and its low-dimensional spectral realizations. (The 3.327-realization is only pseudo-classical: none of its faces are regular.)

U_{51} Small Dodecahemidodecahedron $[5/4, 5|5]$
U_{70} Great Dodecahemidodecahedron $[5/3, 5/2|5/3]$ skel $\cong U_{24}, U_{54}, U_{49}, U_{71}$

λ	Dim	Min Poly	#V: 30	V Fig: $(5.10)^2$	Description
4	1	$x-4$	1	$(5.10)^2/0$	dot
$1+\sqrt{5}\approx 3.236$ $1-\sqrt{5}\approx -1.236$	3	x^2-2x-4	30	5.10.5/4.10 (5/2.10/3.5/3.10/3	U_{51} (co I) U_{70} (co I)
-1	4	$x+1$	5	-	- (sup I)
1	4	$x-1$	30	-	- (sub I)
2	5	$x-2$	15	-	-
-2	10	$x+2$	30	-	-



(a) Skeleton (b) $\lambda = 4$ (c) $\lambda = 1+\sqrt{5} : U_{51}$ (d) $\lambda = 1-\sqrt{5} : U_{70}$

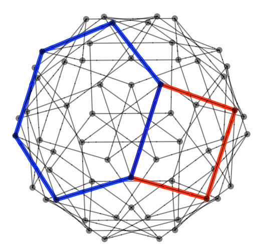
FIGURE 47. The U_{51} / U_{70} skeleton (a) and its low-dimensional spectral realizations. (The $(1+\sqrt{5})$- and $(1-\sqrt{5})$-realizations are classical.)

U_{52} : see U_{23}
U_{53} : see U_{22}
U_{54} : see U_{24}
U_{55} : see U_{25}

U_{56} Rhombicosahedron $[2, 5/2, 3|]$ skel $\cong U_{38}, U_{44}$

λ	Dim	Min Poly	#V: 60	V Fig: $(4.6)^2$	Description
4	1	$x-4$	1	$(4.6)^2/0$	dot
$-1-\sqrt{5} \approx -3.236$ $-1+\sqrt{5} \approx 1.236$	4	x^2+2x-4	- -	- -	- -
-2	4	$x+2$	-	-	-
-3	5	$x+3$	-	-	-
$\frac{1}{2}\left(1-\sqrt{17}\right) \approx -1.561$ $\frac{1}{2}\left(1+\sqrt{17}\right) \approx 2.561$	5	x^2-x-4	- -	- -	- -
-1	6	$x+1$	-	-	-
3	6	$x-3$	-	-	-
0	10	x	-	-	-
1	10	$x-10$	-	-	-

(a) Skeleton



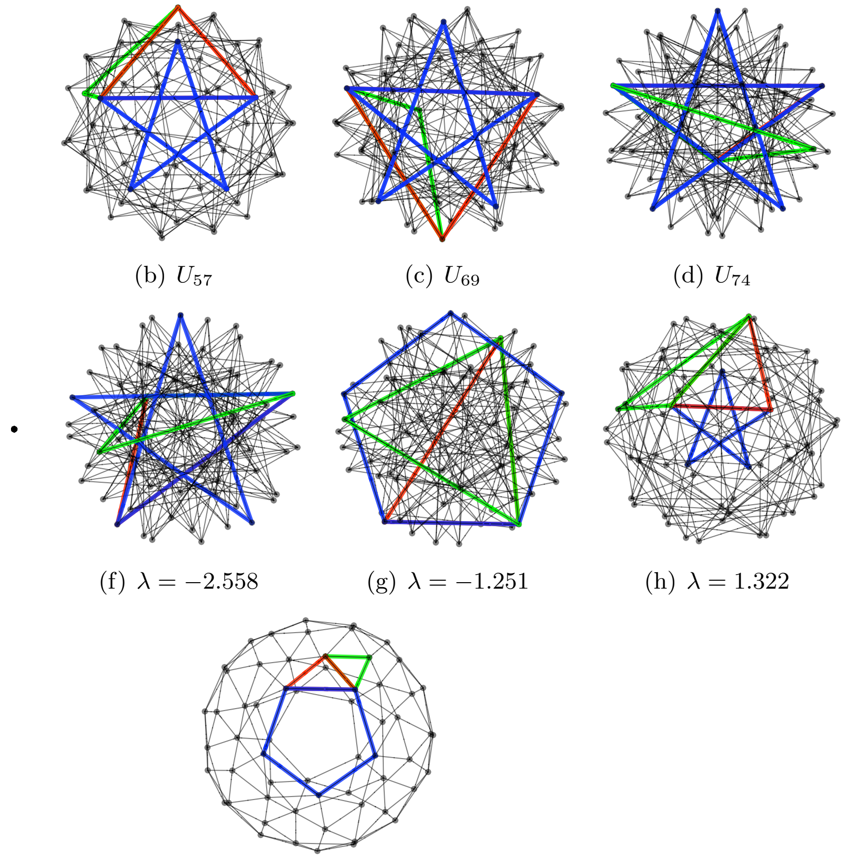
(b) U_{56}

(c) $\lambda = 4$

FIGURE 48. The U_{56} skeleton (a), its classical realization (b), and its low-dimensional spectral realization.

U_{57}	Great Snub Icosidodecahedron	$[|2, 5/2, 3]$
U_{69}	Great Inverted Snub Icosidodecahedron	$[|5/3, 2, 3]$
U_{74}	Great Retrosnub Icosidodecahedron	$[|3/2, 5/3, 2]$

λ	Dim	Min Poly	#V: 60	V Fig: $3^4.5$	Desc
5	1	$x-5$	1	$(3^4.5)/0$	dot
-2.558	3	$x^4-2x^3-13x^2+4x+19$	-	-	-
-1.251			-	-	-
1.322			-	-	-
4.487			-	-	-
-2.280	4	$x^4-8x^2-2x+10$	-	-	-
-1.507			-	-	-
1.070			-	-	-
2.716			-	-	-
-2.351	5	$x^5+x^4-11x^3-19x^2-x+1$	-	-	-
-2.135			-	-	-
-0.285			-	-	-
0.195			-	-	-
3.576			-	-	-
-1	6	$x+1$	-	-	-



(a) Skeleton (b) U_{57} (c) U_{69} (d) U_{74}

(e) $\lambda = 5$ (f) $\lambda = -2.558$ (g) $\lambda = -1.251$ (h) $\lambda = 1.322$

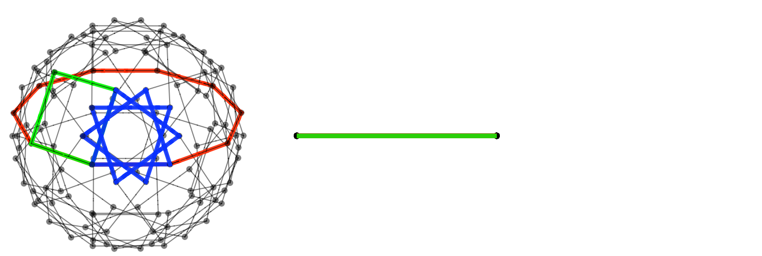
(i) $\lambda = 4.487$

FIGURE 49. The U_{57} / U_{69} / U_{74} skeleton (a), its classical realizations (b-d), and its low-dimensional spectral realizations.

U_{58} : see U_{37}

U_{59} Truncated Dodecadodecahedron $[5/3, 2, 5|]$

λ	Dim	Min Poly	#V: 120	V Fig: 4.10^2	Desc
-3	1	$x+3$	1	$(4.10^2)/0$	dot
3	1	$x-3$	-	-	-
-2.170 -0.311 1.481	4	x^3+x^2-3x-1	- - -	- - -	- - -
-1.481 0.311 2.170	4	x^3-x^2-3x+1	- - -	- - -	- - -
-2.681 -0.642 2.323	5	x^3+x^2-6x-4	- - -	- - -	- - -
-2.323 0.642 2.681	5	x^3-x^2-6x+4	- - -	- - -	- - -
-2	5	$x+2$	-	-	-
2	5	$x-2$	-	-	-
$-1-\sqrt{3}\approx-2.732$ $-1+\sqrt{3}\approx 0.732$	6	x^2+2x-2	- -	- -	- -
$1-\sqrt{3}\approx-0.732$ $1+\sqrt{3}\approx 2.732$	6	x^2-2x-2	- -	- -	- -
-1	10	$x+1$	-	-	-
0	10	x	-	-	-
1	10	$x-1$	-	-	-



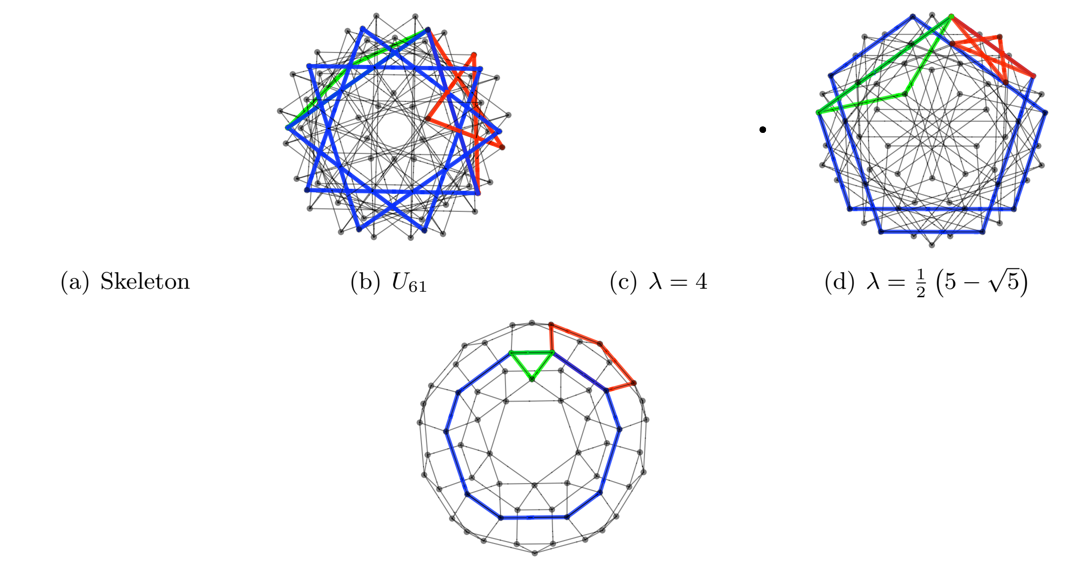
(a) Skeleton (b) U_{59} (c) $\lambda=-3$ (d) $\lambda=3$

FIGURE 50. The U_{59} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations.

U_{60} : see U_{40}

U_{61} Great Dodecicosidodecahedron $[5/2, 3|5/3]$ skel $\cong U_{67}$, U_{73}

λ	Dim	Min Poly	#V: 60	V Fig: 3.10.5.10	Desc
4	1	$x-4$	1	(3.10.5.10)/0	dot
$\frac{1}{2}\left(5-\sqrt{5}\right) \approx 1.381$ $\frac{1}{2}\left(5+\sqrt{5}\right) \approx 3.618$	3	x^2-5x+5	- -	- -	- -
$-\sqrt{5} \approx -2.236$ $\sqrt{5} \approx 2.236$	4	x^2-5	- -	- -	- -
-1	4	$x+1$	-	-	-
1	4	$x-1$	-	-	-
-2.477 0.551 2.925	5	x^3-x^2-7x+4	- - -	- - -	- - -
0	6	x	-	-	-
$\frac{1}{2}\left(-3-\sqrt{5}\right) \approx -2.618$ $\frac{1}{2}\left(-3+\sqrt{5}\right) \approx -0.381$	8	x^2+3x+1	- -	- -	- -



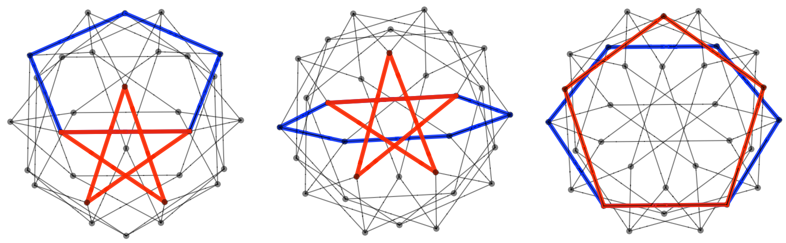
(a) Skeleton (b) U_{61} (c) $\lambda = 4$ (d) $\lambda = \frac{1}{2}\left(5-\sqrt{5}\right)$

(e) $\lambda = \frac{1}{2}\left(5+\sqrt{5}\right)$

FIGURE 51. The U_{61} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations.

U_{62} Small Dodecahemicosahedron $[5/3, 5/2|3]$
U_{65} Great Dodecahemicosahedron $[5/4, 5|3]$ skel $\cong U_{36}$

λ	Dim	Min Poly	#V: 30	V Fig: $(5.6)^2$	Description
4	1	$x-4$	1	$(5.6)^2/0$	dot
-3	4	$x+3$	-	-	-
-1	4	$x+1$	-	-	-
-2	5	$x+2$	-	-	-
0	5	x	-	-	-
2	11	$x-11$	30	-	$\hat{U}_{62}$ / $\hat{U}_{65}$



(a) Skeleton (b) U_{36} (c) $U_{62} : \lambda = 2$ (d) $U_{65} : \lambda = 2$

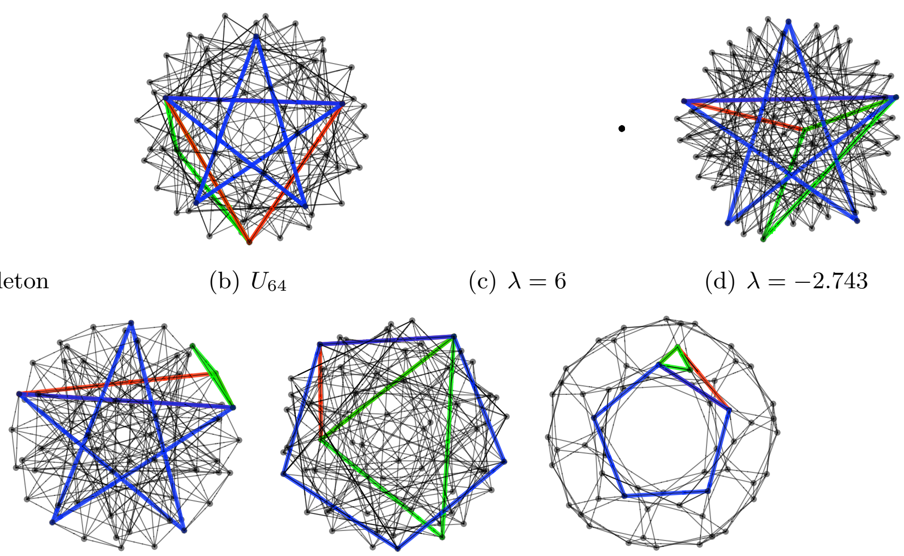
(e) $\lambda = 4$

Figure 52. The U_{62} / U_{65} skeleton (a), its classical realizations (b, c), and its low-dimensional spectral realization. The classical realizations are eigenic (and harmonious), but do not represent a spectral realization. (Evidently, both are projections of the 11-dimensional 2-realization, five-sided faces becoming either pentagons or pentagrams; yet the overall resulting edge set is shared by both projections. Note that U_{36} is also a projection of the high-dimensional form, but doesn't have a companion. Either this is quite remarkable, or I've missed something important.)

U_{63} : see U_{50}

U_{64} Great Snub Dodecicosidodecahedron $[|5/3, 5/2, 3]$

λ	Dim	Min Poly	#V: 60	V Fig: $3^3.5.3.5$	Desc
6	1	$x-6$	1	$(3^3.5.3.5)/0$	dot
-2.743 -0.729 0.393 5.078	3	$x^4-2x^3-15x^2-4x+4$	- - - -	- - - -	- - - -
$\frac{1}{2}\left(-3-\sqrt{5}\right)\approx -2.618$ $\frac{1}{2}\left(-3+\sqrt{5}\right)\approx -0.381$	4	x^2+3x+1	- -	- -	- -
$\frac{1}{2}\left(1-\sqrt{13}\right)\approx -1.302$ $\frac{1}{2}\left(1+\sqrt{13}\right)\approx 2.302$	4	x^2-x-3	- -	- -	- -
$\frac{1}{2}\left(3-\sqrt{17}\right)\approx -0.561$ $\frac{1}{2}\left(3+\sqrt{17}\right)\approx 3.561$	5	x^2-3x-2	- -	- -	- -
-3	10	$x+3$	-	-	-
1	11	$x-1$	-	-	-



(a) Skeleton (b) U_{64} (c) $\lambda = 6$ (d) $\lambda = -2.743$

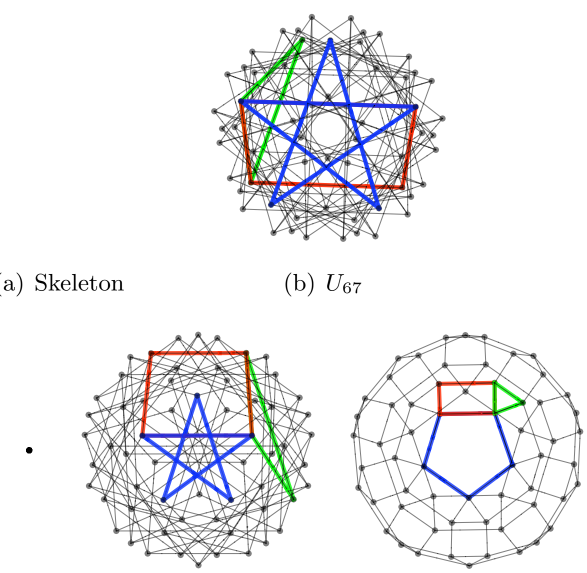
(e) $\lambda = -0.729$ (f) $\lambda = 0.393$ (g) $\lambda = 5.078$

FIGURE 53. The U_{64} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations.

U_{65} : see U_{62}
U_{66} : see U_{26}

U_{67} Uniform Great Rhombicosidodecahedron $[5/3, 3|2]$ skel $\cong U_{61}, U_{73}$

λ	Dim	Min Poly	#V: 60	V Fig: 3.10.5.10	Description
4	1	$x-4$	1	(3.10.5.10)/0	dot
$\frac{1}{2}\left(5-\sqrt{5}\right) \approx 1.381$ $\frac{1}{2}\left(5+\sqrt{5}\right) \approx 3.618$	3	x^2-5x+5	- -	- -	- -
$-\sqrt{5} \approx -2.236$ $\sqrt{5} \approx 2.236$	4	x^2-5	- -	- -	- -
-1	4	$x+1$	-	-	-
1	4	$x-1$	-	-	-
-2.477 0.551 2.925	5	x^3-x^2-7x+4	- - -	- - -	- - -
0	6	x	-	-	-
$\frac{1}{2}\left(-3-\sqrt{5}\right) \approx -2.618$ $\frac{1}{2}\left(-3+\sqrt{5}\right) \approx -0.381$	8	x^2+3x+1	- -	- -	- -



(a) Skeleton (b) U_{67}

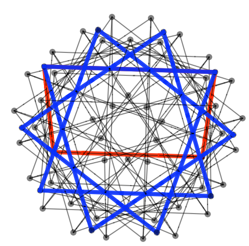
(c) $\lambda = 4$ (d) $\lambda = \frac{1}{2}\left(5-\sqrt{5}\right)$ (e) $\lambda = \frac{1}{2}\left(5+\sqrt{5}\right)$

FIGURE 54. The U_{67} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations.

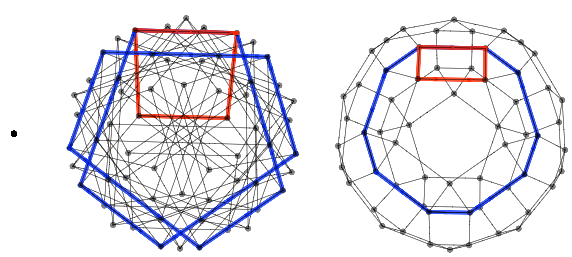
U_{68} : see U_{28}
U_{69} : see U_{57}
U_{70} : see U_{51}
U_{71} : see U_{49}
U_{72} : see U_{32}

U_{73} Great Rhombidodecahedron $[3/2, 5/3, 2|]$ skel $\cong U_{61}, U_{67}$

λ	Dim	Min Poly	#V: 60	V Fig: $(4.10)^2$	Description
4	1	$x-4$	1	$(4.10)^2/0$	dot
$\frac{1}{2}\left(5-\sqrt{5}\right) \approx 1.381$ $\frac{1}{2}\left(5+\sqrt{5}\right) \approx 3.618$	3	x^2-5x+5	- -	- -	- -
$-\sqrt{5} \approx -2.236$ $\sqrt{5} \approx 2.236$	4	x^2-5	- -	- -	- -
-1	4	$x+1$	-	-	-
1	4	$x-1$	-	-	-
-2.477 0.551 2.925	5	x^3-x^2-7x+4	- - -	- - -	- - -
0	6	x	-	-	-
$\frac{1}{2}\left(-3-\sqrt{5}\right) \approx -2.618$ $\frac{1}{2}\left(-3+\sqrt{5}\right) \approx -0.381$	8	x^2+3x+1	- -	- -	- -



(a) Skeleton (b) U_{73}



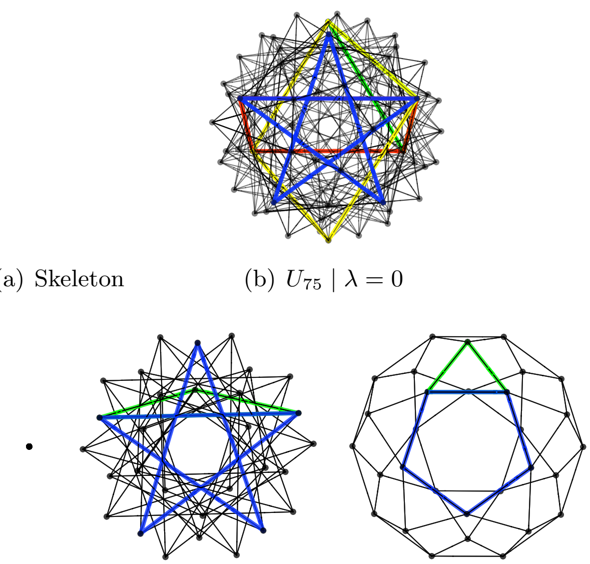
(c) $\lambda = 4$ (d) $\lambda = \frac{1}{2}\left(5-\sqrt{5}\right)$ (e) $\lambda = \frac{1}{2}\left(5+\sqrt{5}\right)$

FIGURE 55. The U_{73} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations.

U_{74} : see U_{57}

U_{75} Great Dirhombicosidodecahedron $[|3/2, 5/3, 3, 5/2]$

λ	Dim	Min Poly	#V: 60	V Fig: $(4.5.4.3)^2$	Desc
8	1	$x-8$	1	$(4.5.4.3)^2/0$	dot
$2\left(1-\sqrt{5}\right) \approx -2.472$ $2\left(1+\sqrt{5}\right) \approx 6.472$	3	$x^2-4x-16$	- -	- -	- -
-2	4	$x+2$	-	-	-
2	4	$x-2$	-	-	-
4	5	$x-4$	-	-	-
-4	10	$x+4$	-	-	-
0	30	x	60	-	$\hat{U}_{75}$



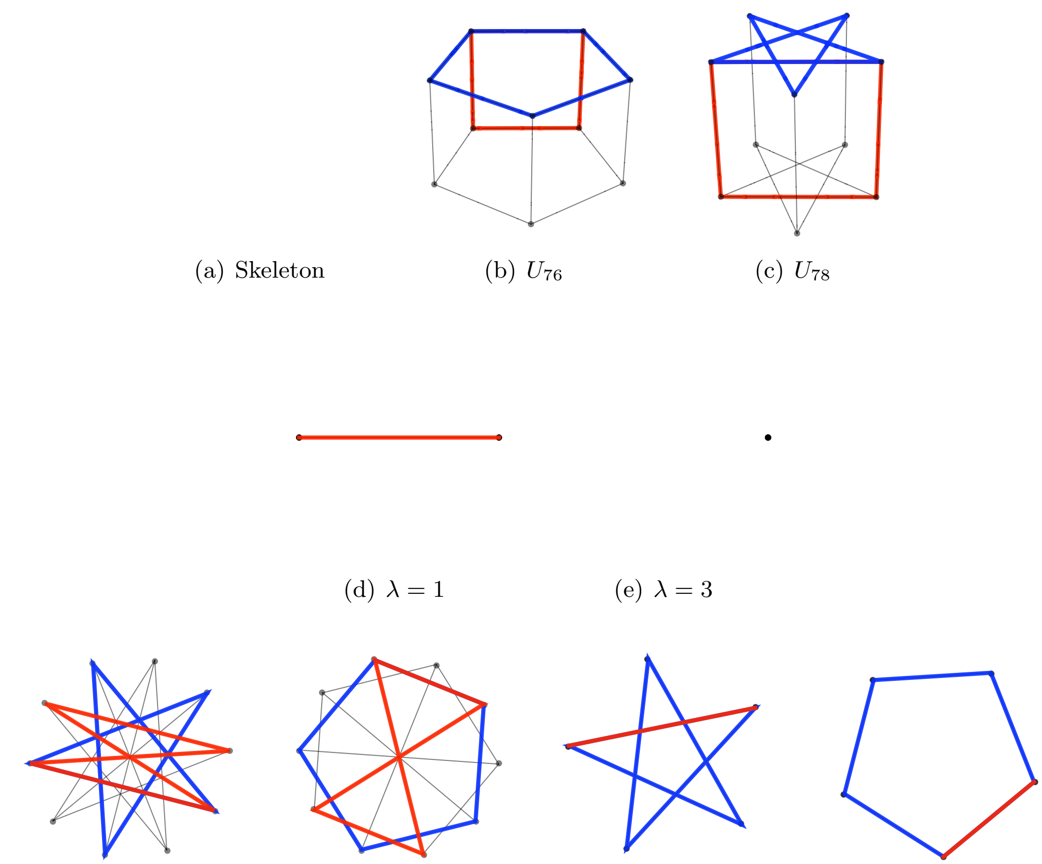
(a) Skeleton (b) $U_{75} \mid \lambda = 0$

(c) $\lambda = 8$ (d) $\lambda = 2\left(1-\sqrt{5}\right)$ (e) $\lambda = 2\left(1+\sqrt{5}\right)$

FIGURE 56. The U_{75} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations. The classical realization is eigenic (and harmonious), but is not a spectral realization.

U_{76}	Polygonal (e.g., Pentagonal) Prism	$[2, 5\|2]$
U_{78}	Polygrammatic (e.g., Pentagrammatic) Prism	$[2, 5/2\|2]$

λ	Dim	Min Poly	#V: 10	V Fig: 5.4^2	Description
1	1	$x-1$	1	$(5.4^2)/0$	dot
3	1	$x-3$	-	-	-
$\frac{1}{2}\left(-3-\sqrt{5}\right) \approx -2.618$ $\frac{1}{2}\left(-3+\sqrt{5}\right) \approx -0.381$	2	x^2+3x+1	- -	- -	- -
$\frac{1}{2}\left(1-\sqrt{5}\right) \approx -0.618$ $\frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.618$	2	x^2-x-1	- -	- -	- -



(a) Skeleton (b) U_{76} (c) U_{78}

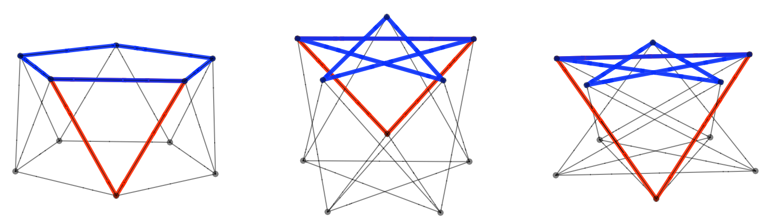
(d) $\lambda = 1$ (e) $\lambda = 3$

(f) $\lambda = \frac{1}{2}\left(-3-\sqrt{5}\right)$ (g) $\lambda = \frac{1}{2}\left(-3+\sqrt{5}\right)$ (h) $\lambda = \frac{1}{2}\left(1-\sqrt{5}\right)$ (i) $\lambda = \frac{1}{2}\left(1+\sqrt{5}\right)$

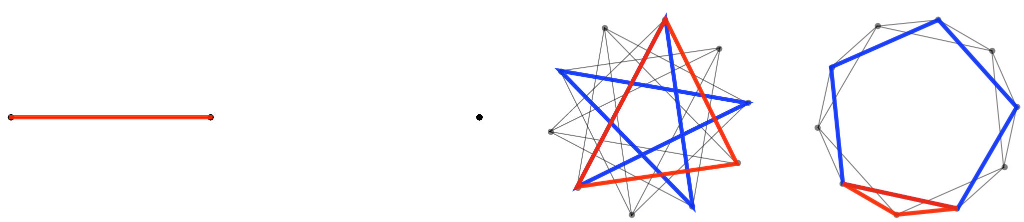
FIGURE 57. The U_{76} / U_{78} skeleton (a), its classical realizations (b, c), and its low-dimensional spectral realizations.

U_{77}	Polygonal (e.g., Pentagonal) Antiprism	$[\|2,2,5]$
U_{79}	Polygrammatic (e.g., Pentagrammatic) Antiprism	$[\|2,2,5/2]$
U_{80}	Polygrammatic (e.g., Pentagrammatic) Crossed Antiprism	$[\|2,2,5/3]$

λ	Dim	Min Poly	#V: 10	V Fig: 5.3^3	Description
0	1	x	1	$(5.3^3)/0$	dot
4	1	$x-4$	-	-	-
$-\sqrt{5}\approx-2.236$ $\sqrt{5}\approx 2.236$	2	x^2-5	- -	- -	- -
-1	4	$x+1$	-	-	-



(a) Skeleton (b) U_{77} (c) U_{79} (d) U_{80}



(e) $\lambda=0$ (f) $\lambda=4$ (g) $\lambda=-\sqrt{5}$ (h) $\lambda=\sqrt{5}$

FIGURE 58. The U_{77} / U_{79} / U_{80} skeleton (a), its classical realizations (b-d), and its low-dimensional spectral realizations.

U_{78} : see U_{76} U_{79} : see U_{77} U_{80} : see U_{77}

4.2.2. *The Uniform-Dual Polyhedra.* **This data is preliminary and incomplete.**

This subsection documents the spectral realizations of *combinatorial duals* [2] of the uniform polyhedra. The data is organized by the standard indices, from dU_1 (tetrahedron, its own dual) to dU_{75} (great dirhombicosidodecacron, the dual of the great dirhombicosidodecahedron), along with the infinite family of prismatic figures U_{76} through U_{80}.

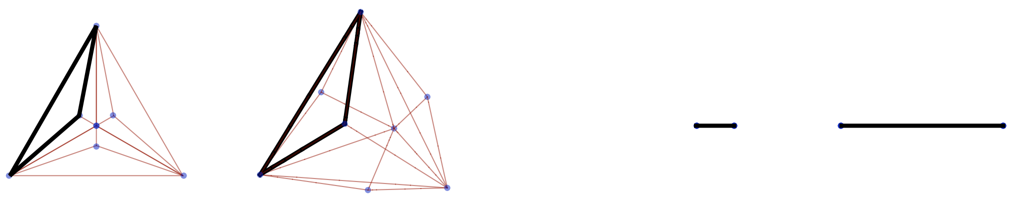
In many cases, a skelton's spectral family includes a representative evocative of a classical form; in the tabulated data, such a realization is indicated by a tilde-topped symbol (e.g., $\widetilde{dU}_2$, suggesting an *approximate* dU_2).

The figures depict a polyhedron's skeleton and its the "low" (≤ 3)-dimensional realizations (captioned by the corresponding eigenvalues). In each case, a representative face cycle has been highlighted. When a polyhedron's "classical" form does not appear in the spectral family, that form is shown separately.

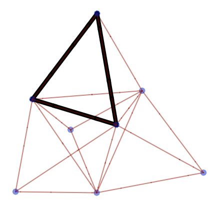
$$\boxed{dU_1 = U_1}$$

dU_2 Triakis Tetrahedron

λ	Dim	Min Poly	#V: 8	Face	Description
$\frac{3}{2}\left(1+\sqrt{5}\right) \approx 4.854$	1	x^2-3x-9	2	-	-
$\frac{3}{2}\left(1-\sqrt{5}\right) \approx -1.854$				-	-
$\frac{1}{2}\left(-1+\sqrt{5}\right) \approx 0.618$	3	x^2+x-1	8	-	$\tilde{dU}_2$
$\frac{1}{2}\left(-1-\sqrt{5}\right) \approx -1.618$				-	-



(a) Skeleton (b) dU_2 (c) $\lambda = \frac{3}{2}\left(1+\sqrt{5}\right)$ (d) $\lambda = \frac{3}{2}\left(1-\sqrt{5}\right)$



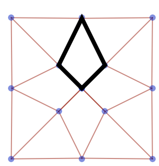
(e) $\lambda = \frac{1}{2}\left(-1+\sqrt{5}\right) : dU_2$ (f) $\lambda = \frac{1}{2}\left(-1-\sqrt{5}\right)$

FIGURE 59. The dU_2 skeleton (a), its classical realization (b), and its spectral realizations.

dU_3 : forthcoming
dU_4 : forthcoming
$dU_5 = U_6$
$dU_6 = U_5$

dU_7 Rhombic Dodecahedron

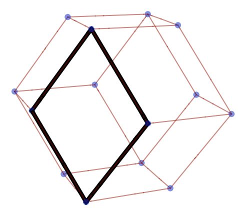
λ	Dim	Min Poly	#V: 14	Face	Description
$2\sqrt{3} \approx 3.464$ $2\sqrt{3} \approx -3.464$	1	$x^2 - 12$	2	- -	- -
2	3	$x - 2$	14	-	dU_7 (co I)
-2	3	$x + 2$	14	-	(co I)
0	6	x	11	-	-



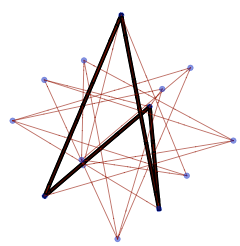
(a) Skeleton

(b) $\lambda = 2\sqrt{3}$

(c) $\lambda = -2\sqrt{3}$



(d) $\lambda = 2$

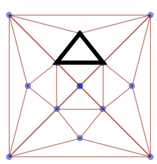


(e) $\lambda = -2 : dU_7$

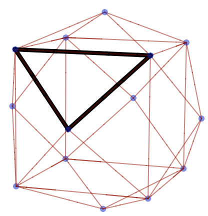
FIGURE 60. The dU_7 skeleton (a) and its low-dimensional spectral realizations.

dU_8 Tetrakis Hexahedron

λ	Dim	Min Poly	#V: 14	Face	Description
$\frac{1}{2}\left(3+\sqrt{57}\right) \approx 5.274$ $\frac{1}{2}\left(3-\sqrt{57}\right) \approx -2.274$	1	$x^2-3x-12$	2	- -	- -
-3	1	$x+3$	3	-	-
0	2	x	4	-	-
$\frac{1}{2}\left(1+\sqrt{17}\right) \approx 2.561$ $\frac{1}{2}\left(1-\sqrt{17}\right) \approx -1.561$	3	x^2-x-4	14	- -	$d\tilde{U}_8$ -
-1	3	$x+1$	5	-	-



(a) Skeleton

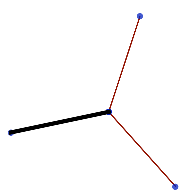


(b) dU_8

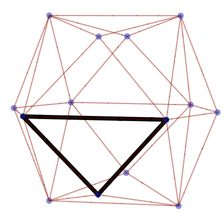
(c) $\lambda = \frac{1}{2}\left(3+\sqrt{57}\right)$

(d) $\lambda = \frac{1}{2}\left(3-\sqrt{57}\right)$

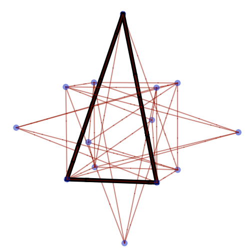
(e) $\lambda = -3$



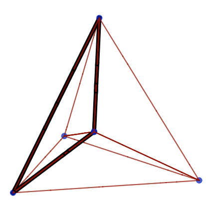
(f) $\lambda = 0$



(g) $\lambda = \frac{1}{2}\left(1+\sqrt{17}\right) : d\tilde{U}_8$



(h) $\lambda = \frac{1}{2}\left(1-\sqrt{17}\right)$

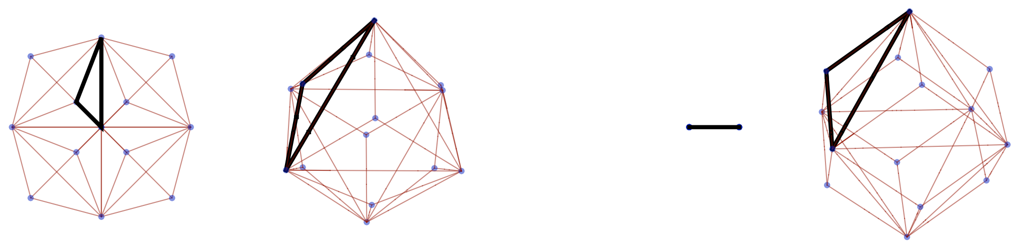


(i) $\lambda = -1$

FIGURE 61. The dU_8 skeleton (a), its classical realization (b), and its spectral realizations. (The $\frac{1}{2}\left(1+\sqrt{17}\right)$-realization is only pseudo-classical.)

dU_9 Small Triakis Octahedron

λ	Dim	Min Poly	#V: 14	Face	Description
6	1	$x-6$	2	-	-
2	3	$x-2$	14	-	$\tilde{dU}_9$
0	4	x	9	-	-
-2	6	$x+2$	14	-	-

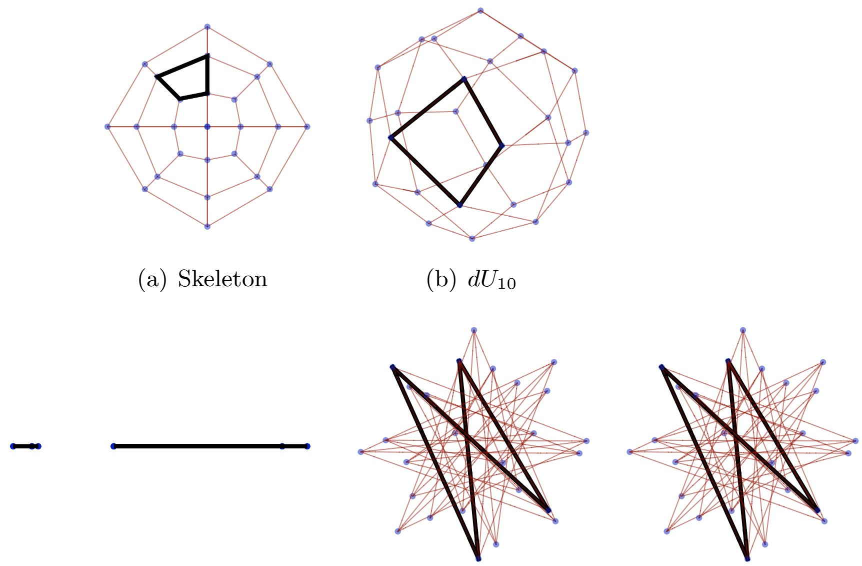


(a) Skeleton (b) dU_9 (c) $\lambda = 6$ (d) $\lambda = 2 : \tilde{dU}_9$

FIGURE 62. The dU_9 skeleton (a), its classical realization (b), and its ($d \leq 3$)-dimensional spectral realizations. (The 2-realization is only pseudo-classical.)

dU_{10} Deltoidal Icositetrahedron

λ	Dim	Min Poly	#V: 26	Face	Description
$\sqrt{14} \approx 3.741$ $-\sqrt{14} \approx -3.741$	1	$x^2 - 14$	3	- -	- -
$2\sqrt{2} \approx 2.828$ $-2\sqrt{2} \approx -2.828$	3	$x^2 - 8$	26	- -	$d\tilde{U}_{10}$ (co I) - (co I)
$\sqrt{2} \approx 1.414$ $-\sqrt{2} \approx -1.414$	5	$x^2 - 2$	13	- -	- -
0	8	x	26	-	-



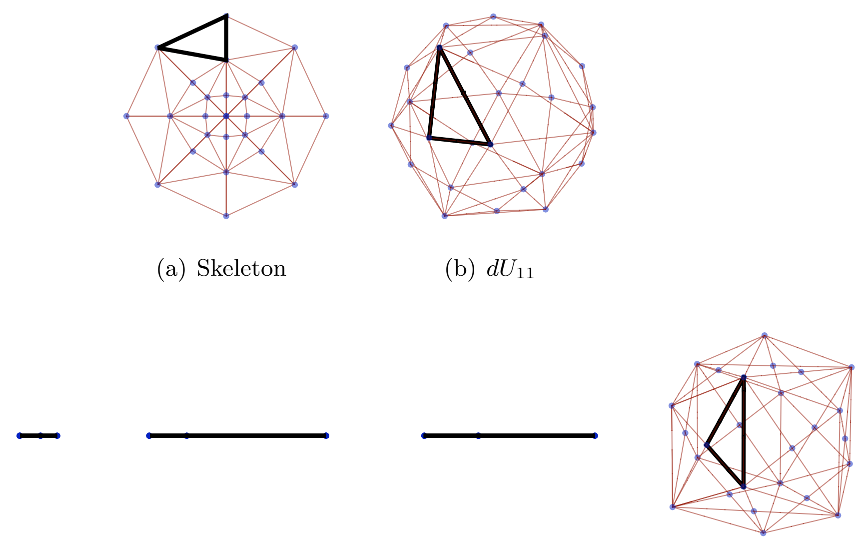
(a) Skeleton (b) dU_{10}

(c) $\lambda = \sqrt{14}$ (d) $\lambda = -\sqrt{14}$ (e) $\lambda = 2\sqrt{2} : d\tilde{U}_{10}$ (f) $\lambda = -2\sqrt{2}$

FIGURE 63. The dU_{10} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations.

dU_{11} Disdyakis Dodecahedron

λ	Dim	Min Poly	#V: 26	Face	Description
5.848 -2.337 -3.511	1	$x^3 - 26x - 48$	3	- - -	- - -
4	3	$x - 4$	26	-	$d\tilde{U}_{11}$
0	4	x	15	-	-
$\sqrt{2} \approx 1.414$ $-\sqrt{2} \approx -1.414$	5	$x^2 - 2$	13	- -	- -
-2	6	$x + 2$	26	-	-



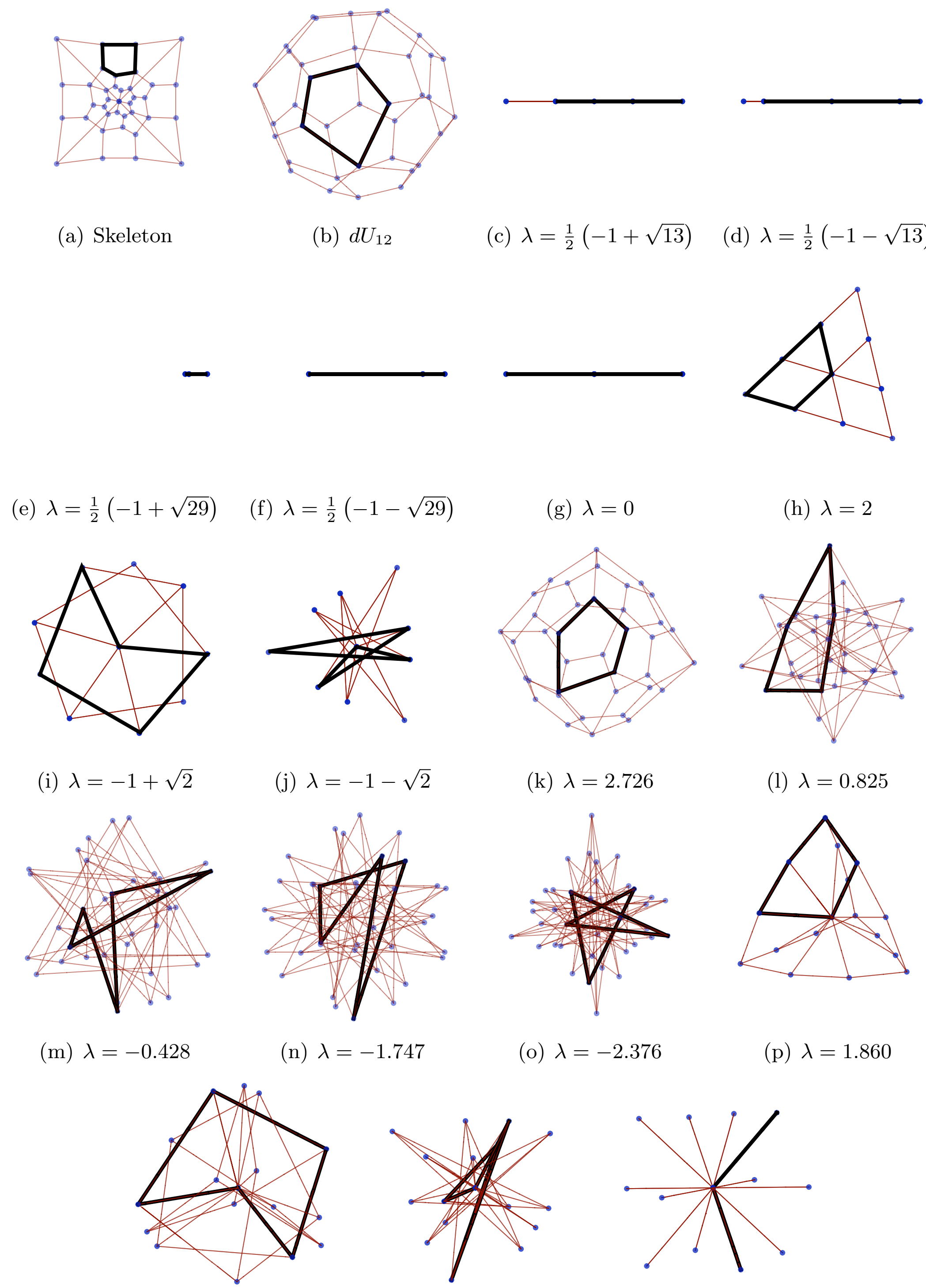
(a) Skeleton (b) dU_{11}

(c) $\lambda = 5.848$ (d) $\lambda = -2.337$ (e) $\lambda = -3.511$ (f) $\lambda = 4 : d\tilde{U}_{11}$

FIGURE 64. The dU_{11} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations.

dU_{12} Pentagonal Icositetrahedron

λ	Dim	Min Poly	#V: 38	Face	Description
$\frac{1}{2}\left(-1+\sqrt{13}\right) \approx 1.302$ $\frac{1}{2}\left(-1-\sqrt{13}\right) \approx -2.302$	1	x^2+x-3	5	- -	- -
$\frac{1}{2}\left(-1+\sqrt{29}\right) \approx 3.192$ $\frac{1}{2}\left(-1-\sqrt{29}\right) \approx -2.192$	1	x^2-x-7	3	- -	- -
0	1	x	3	-	-
2	2	$x-2$	10	-	-
$-1+\sqrt{2} \approx 0.414$ $-1-\sqrt{2} \approx -2.414$	2	x^2+2x-1	10	- -	- -
2.726 0.825 -0.428 -1.747 -2.376	3	$x^5+x^4-8x^3-9x^2+7x+4$	38	- - - - -	- - - - -
1.860 0.254 -2.114	3	x^3-4x+1	17	- - -	- - -
1	3	$x-1$	13	-	-



(a) Skeleton (b) dU_{12} (c) $\lambda = \frac{1}{2}\left(-1+\sqrt{13}\right)$ (d) $\lambda = \frac{1}{2}\left(-1-\sqrt{13}\right)$

(e) $\lambda = \frac{1}{2}\left(-1+\sqrt{29}\right)$ (f) $\lambda = \frac{1}{2}\left(-1-\sqrt{29}\right)$ (g) $\lambda = 0$ (h) $\lambda = 2$

(i) $\lambda = -1+\sqrt{2}$ (j) $\lambda = -1-\sqrt{2}$ (k) $\lambda = 2.726$ (l) $\lambda = 0.825$

(m) $\lambda = -0.428$ (n) $\lambda = -1.747$ (o) $\lambda = -2.376$ (p) $\lambda = 1.860$

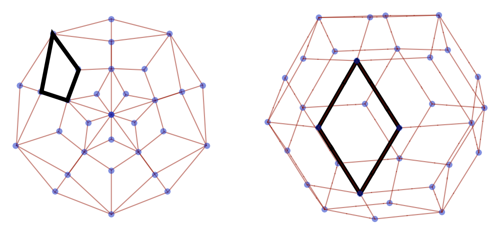
(q) $\lambda = 0.254$ (r) $\lambda = -2.114$ (s) $\lambda = 1$

FIGURE 65. The dU_{12} skeleton (a), its classical realization (b), and its spectral realizations.

$dU_{13} - dU_{21}$: forthcoming
$dU_{22} = U_{23}$
$dU_{23} = U_{22}$

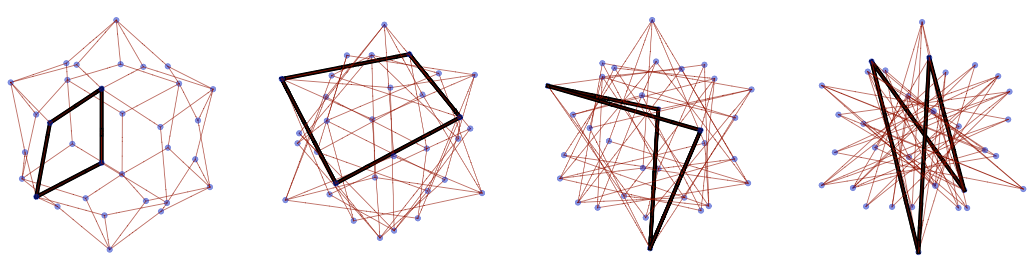
dU_{24} Rhombic Triacontahedron

λ	Dim	Min Poly	#V: 32	Face	Description
$\sqrt{15} \approx 3.872$ $-\sqrt{15} \approx -3.872$	1	$x^2 - 15$	2	- -	- (co I) - (co I)
3.077 0.726 -0.726 -3.077	3	$x^4 - 10x^2 + 5$	32	- - - -	$d\tilde{U}_{24}$ (co II) - (co II) - (co II) - (co II)
$\sqrt{3} \approx 1.732$ $-\sqrt{3} \approx -1.732$	5	$x^2 - 3$	16	- -	- -
0	8	x	21	-	-



(a) Skeleton (b) Classical

(c) $\lambda = \sqrt{15}$ (d) $\lambda = -\sqrt{15}$

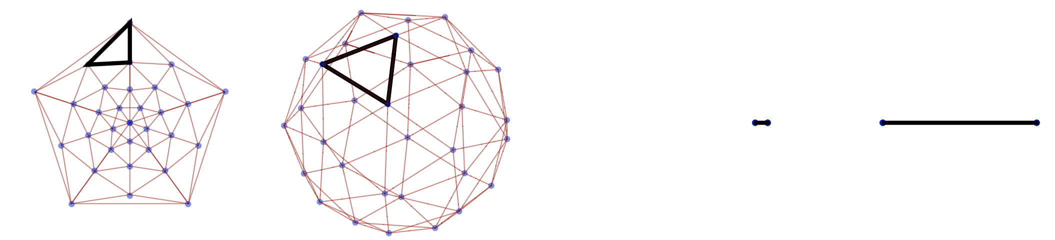


(e) $\lambda = 3.077 : d\tilde{U}_{24}$ (f) $\lambda = 0.726$ (g) $\lambda = -0.726$ (h) $\lambda = -3.077$

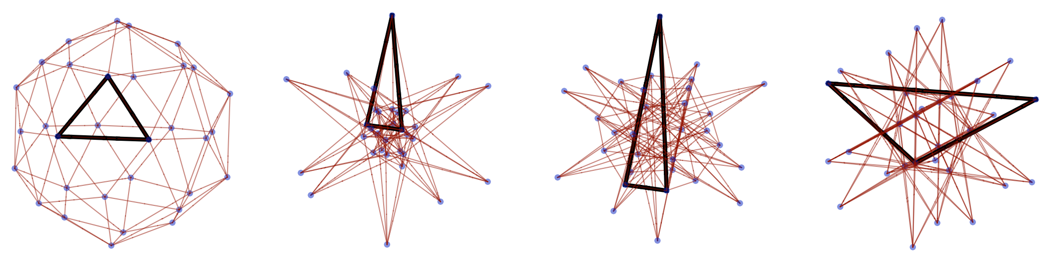
FIGURE 66. The dU_{24} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations.

dU_{25} Pentakis Dodecahedron

λ	Dim	Min Poly	#V: 32	Face	Description
$\frac{1}{2}\left(3+\sqrt{69}\right) \approx 5.653$ $\frac{1}{2}\left(3-\sqrt{69}\right) \approx -2.653$	1	$x^2-3x-15$	2	- -	- -
4.392 0.215 -2.156 -2.451	3	$x^4-15x^2-20x+5$	32	- - - -	$d\tilde{U}_{25}$ - - -
0	4	x	21	-	-
-2	4	$x+2$	11	-	-
$\frac{1}{2}\left(1+\sqrt{13}\right) \approx 2.302$ $\frac{1}{2}\left(1-\sqrt{13}\right) \approx -1.302$	5	x^2-x-3	16	- -	- -



(a) Skeleton (b) dU_{25} (c) $\lambda = \frac{1}{2}\left(3+\sqrt{69}\right)$ (d) $\lambda = \frac{1}{2}\left(3-\sqrt{69}\right)$

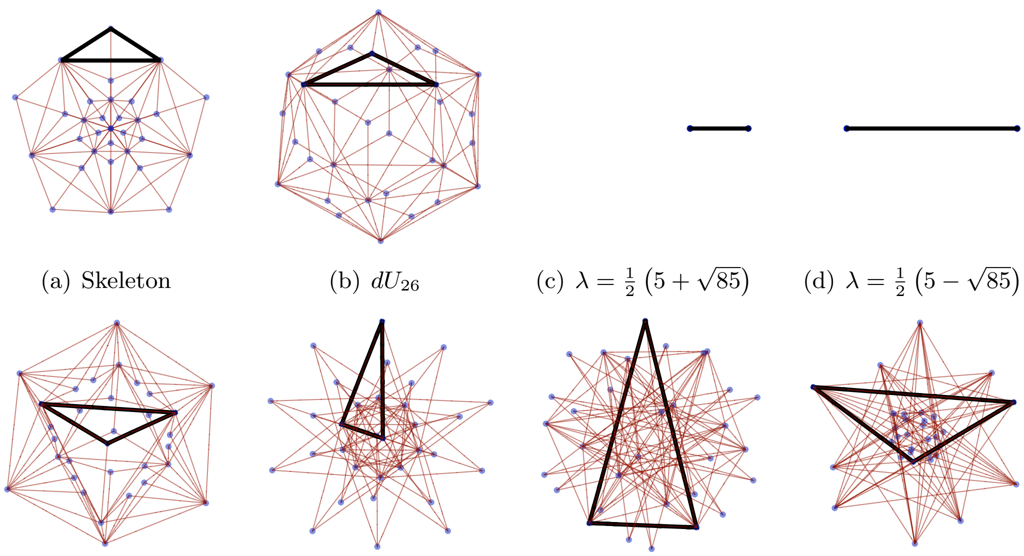


(e) $\lambda = 4.392 : d\tilde{U}_{25}$ (f) $\lambda = 0.215$ (g) $\lambda = -2.156$ (h) $\lambda = -2.451$

FIGURE 67. The dU_{25} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations. (The 4.392-realization is only pseudo-classical.)

dU_{26} Triakis Icosahedron

λ	Dim	Min Poly	#V: 32	Face	Description
$\frac{1}{2}\left(5+\sqrt{85}\right) \approx 7.109$ $\frac{1}{2}\left(5-\sqrt{85}\right) \approx -2.109$	1	$x^2 - 5x - 15$	2	- -	- -
4.392 0.215 -2.156 -2.451	3	$x^4 - 15x^2 - 20x + 5$	32	- - - -	- - - -
$\frac{1}{2}\left(-1+\sqrt{13}\right) \approx 1.302$ $\frac{1}{2}\left(-1-\sqrt{13}\right) \approx -2.302$	5	$x^2 + x - 3$	16	- -	- -
0	8	x	21	-	-



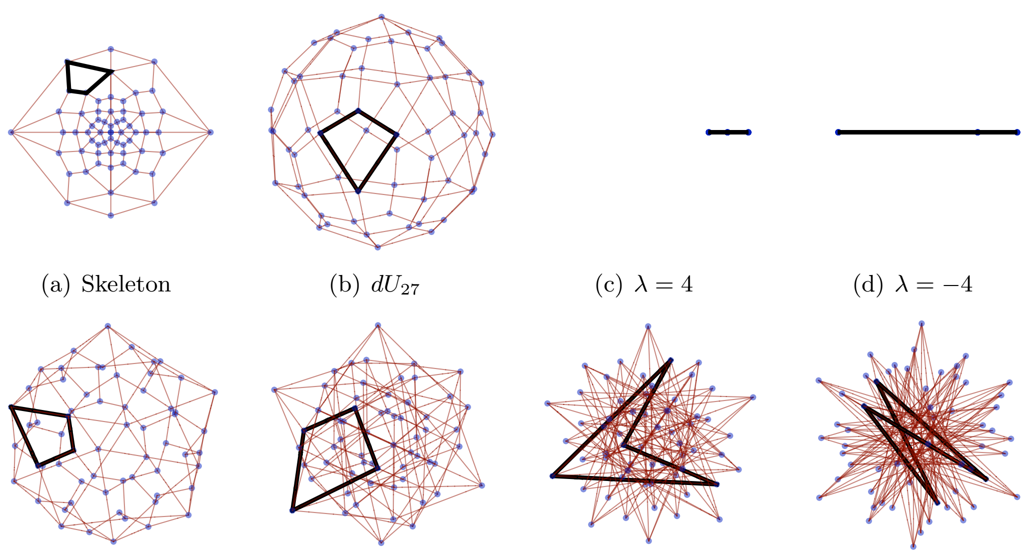
(a) Skeleton (b) dU_{26} (c) $\lambda = \frac{1}{2}\left(5+\sqrt{85}\right)$ (d) $\lambda = \frac{1}{2}\left(5-\sqrt{85}\right)$

(e) $\lambda = 4.392$ (f) $\lambda = 0.215$ (g) $\lambda = -2.156$ (h) $\lambda = -2.451$

FIGURE 68. The dU_{26} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations.

dU_{27} Deltoidal Hexecontahedron

λ	Dim	Min Poly	#V: 62	Face	Description
4	1	$x-4$	3	-	-
-4	1	$x+4$	3	-	-
3.531	3	x^4-16x^2+44	62	-	- (co I)
1.878				-	- (co II)
-1.878				-	- (co II)
-3.531				-	- (co I)
$\sqrt{3}\approx 1.732$	4	x^2-3	51	-	- (co III)
$-\sqrt{3}\approx -1.732$				-	- (co III)
1	4	$x-1$	16	-	-
-1	4	$x+1$	16	-	-
$1+\sqrt{3}\approx 2.732$	5	x^2-2x-2	31	-	-
$1-\sqrt{3}\approx -0.732$				-	-
$-1+\sqrt{3}\approx 0.732$	5	x^2+2x-2	31	-	-
$-1-\sqrt{3}\approx -2.732$				-	-
0	12	x	62	-	-



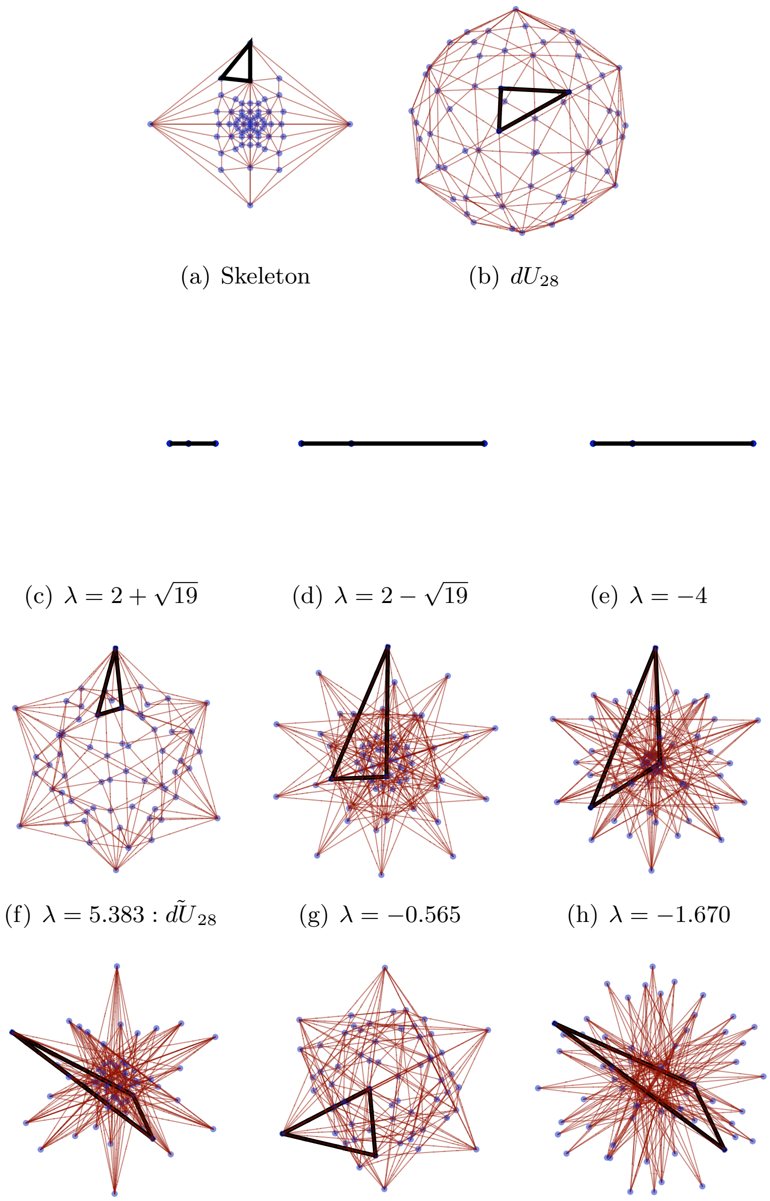
(a) Skeleton (b) dU_{27} (c) $\lambda=4$ (d) $\lambda=-4$

(e) $\lambda=3.531$ (f) $\lambda=1.878$ (g) $\lambda=-1.878$ (h) $\lambda=-3.531$

FIGURE 69. The dU_{27} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations.

dU_{28} Disdyakis Triacontahedron

λ	Dim	Min Poly	#V: 62	Face	Description
$2+\sqrt{19} \approx 6.358$ $2-\sqrt{19} \approx -2.358$	1	$x^2-4x-15$	3	- -	- -
-4	1	$x+4$	3	-	-
5.383 -0.565 -1.670 -3.147	3	$x^4-21x^2-40x-16$	62	- - - -	$d\tilde{U}_{28}$ - - -
$\sqrt{5} \approx 2.236$ $-\sqrt{5} \approx -2.236$	3	x^2-5	62	- -	- -
$\sqrt{3} \approx 1.732$ $-\sqrt{3} \approx -1.732$	4	x^2-3	51	- -	- (co I) - (co I)
1	4	$x-1$	16	-	-
-1	4	$x+1$	16	-	-
$2+\sqrt{3} \approx 3.732$ $2-\sqrt{3} \approx 0.267$	5	x^2-4x+1	31	- -	- -
0	5	x	31	-	-
-2	10	$x+2$	31	-	-



(a) Skeleton (b) dU_{28}

(c) $\lambda = 2 + \sqrt{19}$ (d) $\lambda = 2 - \sqrt{19}$ (e) $\lambda = -4$

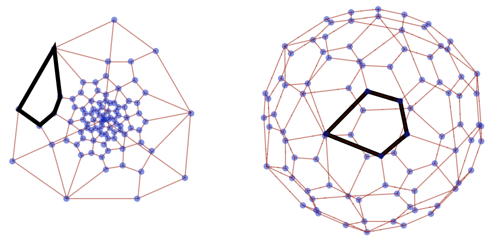
(f) $\lambda = 5.383 : d\tilde{U}_{28}$ (g) $\lambda = -0.565$ (h) $\lambda = -1.670$

(i) $\lambda = -3.147$ (j) $\lambda = \sqrt{5}$ (k) $\lambda = -\sqrt{5}$

FIGURE 70. The dU_{28} skeleton (a) and its low-dimensional spectral realizations.

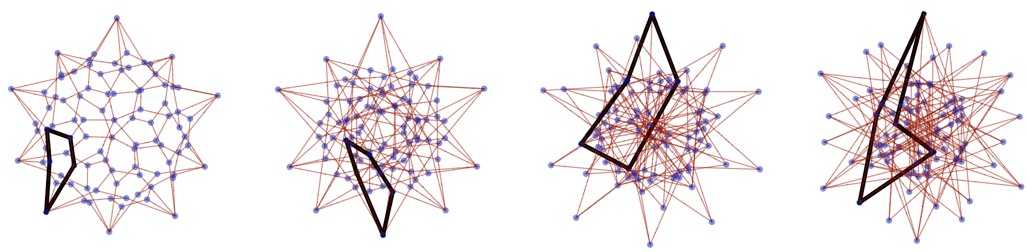
dU_{29} Pentagonal Hexecontahedron

λ	Dim	Min Poly	#V: 92	Face	Description
$\frac{1}{2}\left(1+\sqrt{33}\right) \approx 3.372$	1	$x^2 - x - 8$	3	-	-
$\frac{1}{2}\left(1-\sqrt{33}\right) \approx -2.372$				-	-
3.144	3	$x^{10} + 2x^9 - 17x^8 - 36x^7 + 83x^6 + 182x^5 - 119x^4 - 260x^3 + 60x^2 + 80x - 20$	92	-	
2.342				-	-
1.253				-	-
0.531				-	-
0.260				-	-
-0.746				-	-
-1.440				-	-
-2.220				-	-
-2.489				-	-
-2.635				-	-
2	4	$x - 2$	81	-	-
-2	4	$x + 2$	81	-	-
1.675	4	$x^3 - 4x + 2$	41	-	-
0.539				-	-
-2.214				-	-
2.729	5	$x^6 - 10x^4 + 21x^2 - 4x - 4$	76	-	-
1.428				-	-
0.606				-	-
-0.360				-	-
-1.806				-	-
-2.597				-	-
0	5	x	41	-	-
1	5	$x - 1$	31	-	-

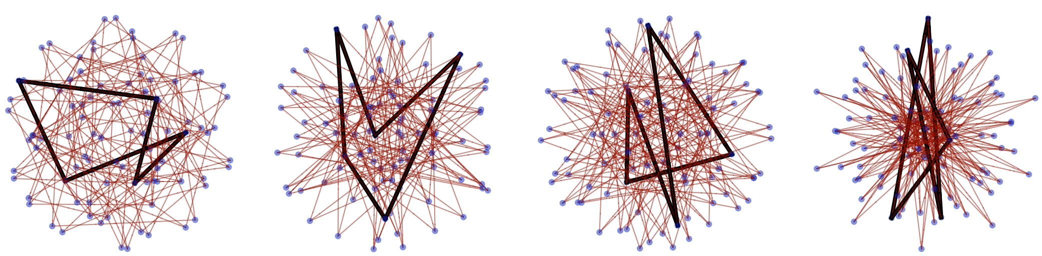


(a) Skeleton (b) dU_{29}

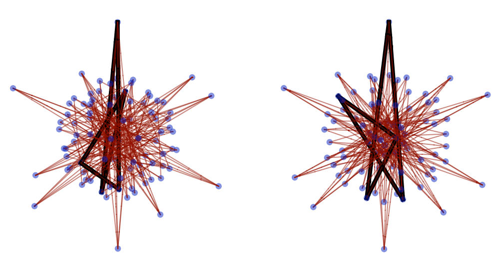
(c) $\lambda = \frac{1}{2}\left(1+\sqrt{33}\right)$ (d) $\lambda = \frac{1}{2}\left(1-\sqrt{33}\right)$



(e) $\lambda = 3.144$ (f) $\lambda = 2.342$ (g) $\lambda = 1.253$ (h) $\lambda = 0.531$



(i) $\lambda = 0.260$ (j) $\lambda = -0.746$ (k) $\lambda = -1.440$ (l) $\lambda = -2.220$



(m) $\lambda = -2.489$ (n) $\lambda = -2.635$

FIGURE 71. The dU_{29} skeleton (a), its classical realization (b), and its low-dimensional spectral realizations.

$$\boxed{dU_{29} - dU_{80} \ : \ \text{forthcoming}}$$

4.3. **Regular Polytopes in 4 and More Dimensions.**

4.3.1. *Four-Dimensional Polytopes.*

$\{3,3,3\}$ 4-Simplex

λ	Dim	Min Poly	#V: 5	Description
4	1	$x-4$	1	dot
-1	4	$x+1$	5	$\{3,3,3\}$

$\{3,3,4\}$ 4-Cross Polytope (24-cell)

λ	Dim	Min Poly	#V (24)	Description
6	1	$x-6$	1	dot
-2	3	$x+2$	4	3-simplex
0	4	x	8	$\{3,3,4\}$

$\{4,3,3\}$ 4-Cube (Hypercube, Tesseract)

λ	Dim	Min Poly	#V: 16	Description
4	1	$x-4$	1	dot
2	4	$x-2$	16	$\{4,3,3\}$
0	6	x	8	skel: cubex
-2	4	$x+2$	16	-
-4	1	$x+4$	2	stick

$\{3,4,3\}$ 24-Cell

λ	Dim	Min Poly	#V: 16	Description
8	1	$x-8$	1	dot
-4	2	$x+4$	3	triangle
4	4	$x-4$	24	$\{3,4,3\}$
-2	8	$x+2$	24	-
0	9	x	12	-

$\{\frac{5}{2},5,3\}$ Small Stellated 120-Cell
$\{5,\frac{5}{2},3\}$ Great Grand 120-Cell

λ	Dim	Min Poly	#V: 120	Description
20	1	$x-20$	1	dot
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\{5,3,3\}$ 120-Cell
$\{5/2,3,3\}$ Great Grand Stellated 120-Cell

λ	Dim	Min Poly	#V: 600	Description
4	1	$x-4$	1	dot
$\frac{1}{2}\left(1+3\sqrt{5}\right)\approx 3.854$ $\frac{1}{2}\left(1-3\sqrt{5}\right)\approx -2.854$	4	x^2-x-11	600	$\{5,3,3\}$ $\{\frac{5}{2},3,3\}$
-1	8	$x+1$	25	-
-2	8	$x+2$	120	-
$\frac{1}{2}\left(5+\sqrt{5}\right)\approx 3.618$ $\frac{1}{2}\left(5-\sqrt{5}\right)\approx 1.381$	9	x^2-5x+5	300	- -
$\frac{1}{2}\left(3+\sqrt{13}\right)\approx 3.302$ $\frac{1}{2}\left(3-\sqrt{13}\right)\approx -0.302$	16	x^2-3x-1	600	- -
$\frac{1}{2}\left(-1+\sqrt{21}\right)\approx 1.791$ $\frac{1}{2}\left(-1-\sqrt{21}\right)\approx -2.791$	16	x^2+x-5	300	- -
0	18	x	60	-
$\sqrt{5}\approx 2.236$ $-\sqrt{5}\approx -2.236$	24	x^2-5	300	- -
$\frac{1}{2}\left(1+\sqrt{5}\right)\approx 1.618$ $\frac{1}{2}\left(1-\sqrt{5}\right)\approx -0.618$	24	x^2-x-1	600	- -
$\frac{1}{2}\left(-3+\sqrt{5}\right)\approx -0.381$ $\frac{1}{2}\left(-3-\sqrt{5}\right)\approx -2.618$	30	x^2+3x+1	300	- -
2.925 -2.477 0.551	25	x^3-x^2-7x+4	300	- - -
2.518 -2.696 1.178	36	x^3-x^2-7x+8	600	- - -
1	40	$x-1$	300	-
$-1+\sqrt{2}\approx 0.414$ $-1-\sqrt{2}\approx -2.414$	48	x^2+2x-1	600	- -

$\{3,3,5\}$	600-Cell
$\{3,3,\frac{5}{2}\}$	Grand 600-Cell
$\{3,5,\frac{5}{2}\}$	Icosahedral 120-Cell
$\{3,\frac{5}{2},5\}$	Great Icosahedral 120-Cell
$\{5,\frac{5}{2},5\}$	Great 120-Cell
$\{\frac{5}{2},5,\frac{5}{2}\}$	Grand Stellated 120-Cell
$\{5,3,\frac{5}{2}\}$	Grand 120-Cell
$\{\frac{5}{2},3,5\}$	Great Stellated 120-Cell

λ	Dim	Min Poly	#V: 120	Description
12	1	$x-12$	1	dot
$3\left(1+\sqrt{5}\right) \approx 9.708$ $3\left(1-\sqrt{5}\right) \approx -3.708$	4	$x^2-6x-36$	120	$\{3,3,5\}$ / $\{5,\frac{5}{2},5\}$ / $\{3,5,\frac{5}{2}\}$ / $\{5,3,\frac{5}{2}\}$ $\{3,3,\frac{5}{2}\}$ / $\{\frac{5}{2},5,\frac{5}{2}\}$ / $\{3,\frac{5}{2},5\}$ / $\{\frac{5}{2},3,5\}$
$2\left(1+\sqrt{5}\right) \approx 6.472$ $2\left(1-\sqrt{5}\right) \approx -2.472$	9	$x^2-4x-16$	60	- -
3	16	$x-3$	120	-
-3	16	$x+3$	60	-
0	25	x	60	-
-2	36	$x+2$	120	-

Polytopes with different cell structures are combined into a single table.

4.3.2. *Higher-Dimensional Regular Polytopes.* There are three classes of regular polytopes in 5 or more dimensions.

$\{3^{d-1}\}$ d-Simplex

λ	Dim	Min Poly	#V: $d+1$	Description
d	1	$x-d$	1	dot
-1	d	$x+1$	$d+1$	$\{3^{d-1}\}$

An exponent indicates repetition; e.g., $\{3^4\} = \{3,3,3,3\}$.

$\{3^{d-2},4\}$ d-Cross Polytope

λ	Dim	Min Poly	#V: $2d$	Description
$2(d-1)$	1	$x-1$	1	dot
0	d	x	$2d$	$\{3^{d-2},4\}$
-2	$d-1$	$x+2$	d	$(d-1)$-simplex

An exponent indicates repetition; e.g., $\{3^3,4\} = \{3,3,3,4\}$.

$\{4, 3^{d-2}\}$ d-Cube

λ	Dim	Min Poly	#V: 2^d	Description
d	1	$x - d$	1	dot
$d - 2$	d	$x - (d - 2)$	2^d	$\{4, 3^{d-2}\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$d - 2i$	$\binom{d}{i}$	$x - (d - 2i)$	$2^{d-(i+1 \bmod 2)}$	$1 \le i < d$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-d$	1	$x + d$	2	stick

An exponent indicates repetition; e.g., $\{4, 3^3\} = \{4, 3, 3, 3\}$.
Justification for these formulas forthcoming.

5. Appendix: Cycle-Decomposition Notation (experimental)

As suggested in the digression in Section 3.2, any not-necessarily-planar n-gon (such as those defining a "face" of our polyhedral realizations) is the *vertex sum* of linear images of the n-cycle's spectral realizations; these are the n-gon's *spectral components*, one for each k from 0 to $\lfloor n/2 \rfloor$, corresponding to the entries in the table of Section 4.1. Also as suggested in that digression, each such linear image is the sum of (in the case of polygons) two similar, but oppositely-oriented, regular polygons. For example, any linear image of the pentagram, $\{5/2\}$, is the vector sum of two pentagrams, a $\{5/2\}$ and an oppositely-oriented $\{5/3\}$ (which is a $\{5/2\}$ whose vertices are traced in reverse order); we call these the *Barlotti components* of the linear image. These facts suggest a potentially useful (though not definitive) way of describing the shapes of cycle realizations.

Definition 3. *An n-gon, P, has* cycle decomposition

$$[n : \alpha_0, \frac{\alpha_1}{\beta_1} c_1, \ldots, \frac{\alpha_m}{\beta_m} c_m] \qquad m := \lfloor n/2 \rfloor \tag{7}$$

Here, $\alpha_k \geq \beta_k$ are the circumradii of the oppositely-oriented $\{n/k\}$ and $\{n/(n-k)\}$ Barlotti components of the k-th spectral component of P; c_k is the cosine *of the angle between the vectors corresponding to the initial vertices in the two Barlotti components. When there is at most one Barlotti component —that is, when $k = 0$ or $n/2$, or when β_k is 0— we suppress the fraction notation (and appended cosine, which we take to be 1):*

$$\frac{\alpha_k}{0} c_k \to \alpha_k \qquad \frac{0}{0} c_k \to 0$$

With this notation, we can retrieve the shape of the k-th spectral component as a planar polygon with vertices $v_0, v_1, \ldots, v_{n-1}$ having coordinates

$$v_j := \left[\begin{array}{c} a_k \cos\left(\frac{\theta_k}{2} + \frac{2\pi jk}{n}\right) + b_k \cos\left(-\frac{\theta_k}{2} - \frac{2\pi jk}{n}\right) \\ a_k \sin\left(\frac{\theta_k}{2} + \frac{2\pi jk}{n}\right) + b_k \sin\left(-\frac{\theta_k}{2} - \frac{2\pi jk}{n}\right) \end{array} \right], \qquad \text{where } \ \theta_\mathrm{k} := \arccos \mathrm{c_k}$$

We cannot retrieve the shape of the original polygon, however, because the notation does not encode how the spectral components are oriented in (multi-dimensional) space relative to one another.

Observe that, because all non-dot spectral realizations of the n-cycle are centered at the origin, the dot component (that is, the 0-th spectral component) of a polygon gives the distance from the polygon's center[13] to the origin. As far as *shape* is concerned, therefore, this component is irrelevant; however, it is convenient for mensuration purposes.

Interestingly, the notations —*including* that 0-th component— for corresponding faces of distinct spectral realizations of a polyhedron are often related in a most uncomplicated way: they are reversals of one another. See, for example, the truncated octahedron (U_8 in Section 4.2.1): for each spectral realization, there is a counterpart (namely, the one with the negative of the first's eigenvalue) such that the cycle-decomposition notations for the hexagonal faces (and the quadrilateral faces) are reversed.

[13] Where the center is defined as the point whose coordinate vector is the average of the coordinate vectors of the figure's vertices.

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