

TETRAHEDRA SHARING VOLUME, FACE-AREAS, AND CIRCUMRADIUS: A HEDRONOMETRIC APPROACH

BLUE, THE HEDRONOMETER
blue@hedronometry.com

ABSTRACT. Volume, face-areas, and circumradius sometimes determine multiple—even infinitely-many— non-isometric tetrahedra. Hedronometry provides a context for unifying and streamlining previous discussions of this fact.

Marcin Mazur [2] posed the following as an open question in 1999:

Mazur’s Question. Is every tetrahedron [...] determined by its volume, the areas of its faces, and the radius of its circumscribed sphere [“Mazur metrics”]?

Within a year, Petr Lisoněk and Robert Israel [1] gave an explicit example of two non-isometric tetrahedra sharing volume $\sqrt{3}/12$, face-areas $\sqrt{7}/4$, $\sqrt{7}/4$, $1/2$, $1/2$, and circumradius (hereafter, simply “radius”) $\sqrt{21}/6$. They refined Mazur’s Question to ask whether the Mazur metrics determine only *finitely many* non-isometric tetrahedra. Later, in 2005, Lu Yang and Zhenbing Zeng [4] exhibited a *continuum* of non-isometric tetrahedra sharing volume 441, areas $84\sqrt{3}$, $63\sqrt{3}$, $63\sqrt{3}$, $63\sqrt{3}$, and radius $43\sqrt{3}/6$.

So, Mazur’s Question has long had its Answer: an emphatic “No”.

This note does not claim to contribute significant new findings on this topic. It primarily revisits existing results in the context of *hedronometry*, the dimensionally-enhanced trigonometry of tetrahedra. For motivation, consider that the Yang-Zeng tetrahedra are parameterized by a portion of a cubic curve in \mathbb{R}^2 with not-immediately-visible symmetry in not-obviously-meaningful parameters x and y :

$$(1) \quad 3(1-x)(17-18y)(1+3x+3y) - 9x^2 - 3x - 37 = 0$$

We can simplify the cubic rather dramatically with the seemingly-arbitrary substitutions $x \rightarrow (23814-h)/23814$ and $y \rightarrow (23814-j)/23814$:

$$(2) \quad hj(56889-h-j) = 4084868810988$$

Even better, introducing an auxiliary parameter k such that $h+j+k = 56889$ (by interesting coincidence, the sum of the squares of the areas), we endow the equation with an impossible-to-miss *three-fold* symmetry in \mathbb{R}^3 :

$$(3) \quad hjk = 4084868810988$$

Best of all, these h , j , k are geometrically *meaningful*: they are the squares of the areas of the tetrahedron’s hedronometric “pseudofaces”, suggesting that hedronometry can provide a useful lens through which to view answers to Mazur’s Question. This note explores that suggestion.

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1. PRELIMINARIES: HEDRONOMETRIC PARAMETERS

A tetrahedron is determined *trigonometrically* by the *lengths* (and arrangement) of its six edges. It is determined *hedronometrically* by the *areas* (and arrangement) of its four faces, W, X, Y, Z , and of its three pseudofaces, H, J, K ; these seven areas exhibit only six degrees of freedom, due to the Sum of Squares identity.

$$(4) \quad W^2 + X^2 + Y^2 + Z^2 = H^2 + J^2 + K^2$$

While pseudofaces have a geometric interpretation,¹ the reader is welcome to take their areas as formally defined by the Law of Opposite Cosines:²

$$(5) \quad \begin{aligned} Y^2 + Z^2 - 2YZ \cos A &= H^2 = W^2 + X^2 - 2WX \cos D \\ Z^2 + X^2 - 2ZX \cos B &= J^2 = W^2 + Y^2 - 2WY \cos E \\ X^2 + Y^2 - 2XY \cos C &= K^2 = W^2 + Z^2 - 2WZ \cos F \end{aligned}$$

where A is the dihedral angle between faces Y and Z , etc. Incidentally, the Law of *Concurrent Cosines*

$$(6) \quad W^2 = X^2 + Y^2 + Z^2 - 2YZ \cos A - 2ZX \cos B - 2XY \cos C$$

gives rise to the Tetrahedron Inequality

$$(7) \quad W \leq X + Y + Z$$

that, for non-zero areas, admits equality only for degenerate “flat” figures with $\sin A = \sin B = \sin C = 0$.

Notation Alert. Almost-every instance we’ll see of areas W, X, Y, Z, H, J, K , volume V , and radius r involves an even power (and sometimes a multiplied constant). To save space and reduce visual clutter, we define:

$$\begin{aligned} w := W^2 \quad x := X^2 \quad y := Y^2 \quad z := Z^2 \quad h := H^2 \quad j := J^2 \quad k := K^2 \\ u := (3V)^2 \quad s := (2r)^2 \end{aligned}$$

Now, a tetrahedron’s volume, V , is given hedronometrically by

$$(8) \quad \begin{aligned} (3V)^4 = u^2 = hjk + 2(wxy + wxz + wyz + xyz) \\ - h(wx + yz) - j(wy + zx) - k(wz + xy) \end{aligned}$$

For the radius, r , we start non-hedronometrically with

$$(9) \quad 16(3V)^2(2r)^2 = 16us = [ad, be, cf]$$

where $\{a, d\}$, $\{b, e\}$, $\{c, f\}$ are pairs of opposite edges, and “[\dots]” is the versatile “Heronic product”³

$$(10) \quad \begin{aligned} [a, b, c] &:= (a + b + c)(-a + b + c)(a - b + c)(a + b - c) \\ &= -a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 \end{aligned}$$

¹Projecting a tetrahedron into a plane parallel to a pair of opposite edges, the quadrilateral whose diagonals are the projections of the chosen edge pair is a “pseudoface” of the figure. For a typical tetrahedron with faces W, X, Y, Z , we take pseudoface H to be associated with the edges common to face-pairs $\{W, X\}$ and $\{Y, Z\}$; pseudoface J , with edges common to face-pairs $\{W, Y\}$ and $\{Z, X\}$; pseudoface K , with edges common to $\{W, Z\}$ and $\{X, Y\}$.

²A catchier name than *Law of Cosines of Dihedral Angles along Opposite Edges*.

³After Heron of Alexandria’s formula, $\frac{1}{4}\sqrt{[a, b, c]}$, for the area of a triangle with sides a, b, c .

With the help of these⁴ relations

$$(11) \quad \begin{aligned} ua^2 &= [H, Y, Z] & ub^2 &= [J, Z, X] & uc^2 &= [K, X, Y] \\ ud^2 &= [H, W, X] & ue^2 &= [J, W, Y] & uf^2 &= [K, W, Z] \end{aligned}$$

we convert (9) to hedronometric form:

$$(12) \quad \begin{aligned} su^3 &= (u^2 - hjk) [H, J, K] + hjk(hj + jk + kh) \\ &+ jk(j - k) \Delta(wx|yz) + kh(k - h) \Delta(wy|zx) + hj(h - j) \Delta(wz|xy) \\ &- h \Delta(wy|zx) \Delta(wz|xy) - j \Delta(wz|xy) \Delta(wx|yz) - k \Delta(wx|yz) \Delta(wy|zx) \end{aligned}$$

with *ad hoc* notation $\Delta(wx|yz) := (w - x)(y - z)$, etc, to save space. Equal-area faces cause (12) to collapse considerably.

1.1. Mazur + Pseudoface = 1. Mazur metrics alone do not necessarily determine a tetrahedron, but Mazur metrics *and a pseudoface* do. This fact will help with enumerating possible solutions to specific cases of Mazur's Question.

To see this, suppose that pseudoface H (with square h) is given, and notice that the Sum of Squares and volume formulas (4) and (8) constitute a quadratic system in j and k that admits two solutions distinguished by a single ambiguous sign:

$$(13) \quad \begin{aligned} 2j &= -h + w + x + y + z - \frac{1}{h} ((w - x)(y - z) \pm \sqrt{\sigma}) \\ 2k &= -h + w + x + y + z + \frac{1}{h} ((w - x)(y - z) \pm \sqrt{\sigma}) \\ \sigma &:= [H, W, X][H, Y, Z] - 4hu^2 \end{aligned}$$

If $\sigma = 0$, then the sign ambiguity is moot and the solutions match.⁵ If the product $(w - x)(y - z)$ vanishes, then the sign ambiguity resolves by swapping pseudofaces J and K in one solution, so that the pair of solutions correspond to isometric tetrahedra; if the product is non-zero, then substituting j and k into the radius formula (12) resolves the sign ambiguity. Therefore, we have shown

The ‘‘Mazur + Pseudoface = 1’’ Principle (MP1).

A tetrahedron is *uniquely determined* (up to isometry) by its face-areas, volume, circumradius, and one pseudoface-area.

In situations where the number of viable values for a pseudoface-area is limited, that number serves as an upper bound on the number of viable tetrahedra.

2. ANSWERING MAZUR'S QUESTION

In the context of Mazur's Question, we consider face-areas W, X, Y, Z (and their squares, w, x, y, z) to be given, so that our tetrahedra are parameterized by pseudoface-areas H, J, K (equivalently, their squares, h, j, k). Ignoring trivial degeneracies⁶ throughout, Mazur (hedrono)metric relations (4), (8), (12) comprise

⁴Here, a is the edge common to faces Y and Z (and also a diagonal of pseudoface H), etc.

⁵One can show that $H = \frac{1}{2}ad \sin \theta_H$, where θ_H is the angle between the vectors along edges a and d . With (11), we have $\sigma = u^2 a^2 d^2 \cos^2 \theta_H$, so $\sigma = 0$ indicates that those edges are orthogonal.

⁶We assume that volume is strictly positive. This implies that face-areas and radius are also strictly positive, and that the dihedral angles in (5) and (6) are strictly between 0 and π . These restrictions, in turn, imply that the squares of the pseudoface-areas are strictly positive. (In certain contexts, pseudoface-area can be *signed*, but that's not a consideration here.)

a system of three polynomial equations in the three parameters. We can eliminate j and k from the system—substituting from (13) into (12), and squaring to release σ from its radical—to arrive at a nonic polynomial whose h -roots correspond to ostensible pseudoface-areas; this nonic is the sum of a messy sextic and the product of h with two tame quartics:

$$(14) \quad p_h(w, x, y, z; u; s) :=$$

$$h \cdot (\delta_y h^4 - \bar{\sigma}_{3z} h^3 + \tau \bar{\sigma}_{3z} h^2 - (su^3 + \rho_z (4\delta_z - \tau^2)) h + \sigma_{1y} \bar{\sigma}_{3y})$$

$$\cdot (\delta_z h^4 - \bar{\sigma}_{3y} h^3 + \tau \bar{\sigma}_{3y} h^2 - (su^3 + \rho_y (4\delta_y - \tau^2)) h + \sigma_{1z} \bar{\sigma}_{3z})$$

$$+ \delta_x^2 \left(\begin{array}{l} 4h^6 \bar{\sigma}_{1x} \\ - h^5 (\tau[\sigma_3]^\oplus + 6\delta_y \delta_z + 12\tau \bar{\sigma}_{1x}) \\ + 3h^4 (su^3 + \tau^2 u^2 - [\delta\sigma_1 - \tau^2(\sigma_3 + \bar{\sigma}_{1x}) + \delta\bar{\sigma}_{3x}]^\oplus + 2\tau\delta_y\delta_z) \\ - h^3 \left(6\tau (su^3 + \tau^2 u^2 - [\delta\sigma_1]^\oplus) - [\rho_x \sigma_3 - 3\rho (3\tau^3 + \bar{\sigma}_{2x})]^\oplus \right. \\ \quad \left. + 3\sigma_{1y}\sigma_{1z} - 12\rho_y\rho_z - 8\delta_y\delta_z[\delta]^\oplus + \tau^3\bar{\sigma}_{1x} + 2\bar{\sigma}_{2x}^2 \right) \\ + h^2 \left[\begin{array}{l} -3\delta (su^3 + \rho (4\delta - \tau^2)) + 4\tau (u^4 - 3\rho^2) \\ + \sigma_3 (2\delta - \tau^2) (4\delta - \tau^2) - 2\delta_x^2 (\sigma_3 + 3\tau (\delta - \tau^2)) \end{array} \right]^\oplus \\ - h \left(\begin{array}{l} [\sigma_3]^\oplus (su^3 + \tau^2 u^2 - [\delta\sigma_1]^\oplus) - [\delta\sigma_1 \bar{\sigma}_3]^\oplus \\ - 3\tau^2 \sigma_{1y}\sigma_{1z} + \delta_x^2 (3\delta_y\delta_z + \tau\bar{\sigma}_{3x}) \end{array} \right) \\ + (\sigma_{1y}\sigma_{1z}[\bar{\sigma}_1]^\oplus + \delta_x^2 (su^3 + \tau^2 u^2 - [\delta\sigma_1]^\oplus)) \end{array} \right)$$

The coefficients reduce to manageable size (if minimal scrutability) with the help of a few more *ad hoc* assignments⁷ that we won't see again:

$$(15) \quad \tau := w + x + y + z$$

$$\delta_x := (w - x)(y - z) \quad \rho_x := (wx - yz) ((w + x) - (y + z))$$

$$\delta_y := (w - y)(x - z) \quad \rho_y := (wy - xz) ((w + y) - (x + z))$$

$$\delta_z := (w - z)(x - y) \quad \rho_z := (wz - xy) ((w + z) - (x + y))$$

$$\sigma_{nm} := u^2 + n\rho_m \quad \bar{\sigma}_{nm} := u^2 - n\rho_m \quad [f]^\oplus := f_y + f_z$$

⁷Assignments are such that expressions of the form “ f_y ” and “ f_z ” become equal when $y = z$ or $w = x$ (in which case, the sextic vanishes and the product of quartics becomes a perfect square). The ρ s exhibit cyclic symmetry via $x \rightarrow y \rightarrow z \rightarrow x$, but the δ s do not. The symbols are not quite *arbitrary*, since

$$[H, W, X][H, Y, Z] - 4hu^2 = h^4 - 2h^3\tau + h^2(\tau^2 - 2(\delta_y + \delta_z)) - 2h(\sigma_{1y} + \sigma_{1z}) + \delta_x^2$$

but their purpose here is primarily to condense the polynomial expression, not to suggest any *structure*. (The symbols are subject to various identities, such as $\delta_x \equiv \delta_y - \delta_z$ and $\tau\delta_y \equiv \rho_x - \rho_z$. The reader is invited to manipulate the polynomial into a more-enlightening form, especially one that makes the derivation from (12) self-evident.) Finally, here is an example of the unusual bracket notation: $[\delta\sigma_3]^\oplus = \delta_y\sigma_{3y} + \delta_z\sigma_{3z}$.

Polynomial p_h has degree 9 in h ; degree 8 in each of w, x, y, z ; degree 8 in u ; and degree 2 in s . We easily obtain polynomials p_j and p_k in j and k via the substitutions $h \rightarrow j \rightarrow k$ and $x \rightarrow y \rightarrow z \rightarrow x$.

We can see that, *usually*, the Mazur metrics admit a finite pool of at most nine candidate values for each parameter h, j, k ; by MP1, they determine at most nine tetrahedra. *Unusually*, relations among the metrics can zero-out coefficients—especially leading or trailing ones—and/or facilitate factoring, reducing a polynomial’s effective degree and draining the candidate pool. In the extreme, a clever choice of metrics can make *all* of the coefficients vanish, removing the polynomial constraints on the corresponding pseudoface-area, and raising the possibility of *infinitely-many* Mazur-determined tetrahedra.

We examine a few circumstances, usual and un-, hedronometrically recasting the results of Lisoněk and Israel [1], Yang and Zeng [4], and Tsai [3] as we go.

2.1. The Scaleno-hedral Case (distinct W, X, Y, Z).

No, Mr. Mazur. Typically, your metrics determine up to nine scaleno-hedral tetrahedra. Exceptionally, up to eight or seven. (*Maybe* only ever up to six.)

When no face-areas match, the leading coefficients of the h - j - k polynomials cannot vanish, so none of the polynomials can be identically zero. Thus, there are as many as nine possible values for h (or j , or k), which implies—by MP1—that there are only as many as nine resulting tetrahedra (up to isometry). Yang and Zeng [4] conjectured this bound in 2005, and Tsai [3] first confirmed it in 2015.⁸

Lisoněk and Israel [1] gave scaleno-hedral metrics⁹ that determine six tetrahedra.

$$(16) \quad (w, x, y, z; u; s) = \left(\frac{261}{4}, \frac{125}{4}, 32, 36; 144; 461\right)$$

The corresponding p_h has a full roster of nine strictly-positive h -roots, but only six of them have viable companion j and k values:¹⁰

$$(17) \quad (h, j, k) = \begin{cases} (116 & , & 37.25 & , & 11.25 &) \\ (10.0522\dots & , & 76.9446\dots & , & 77.5030\dots &) \\ (10.9690\dots & , & 104.8131\dots & , & 48.7178\dots &) \\ (45.6808\dots & , & 108.7564\dots & , & 10.0627\dots &) \\ (74.3631\dots & , & 9.8696\dots & , & 80.2672\dots &) \\ (114.5688\dots & , & 12.0394\dots & , & 37.8916\dots &) \end{cases}$$

Tsai reports that numerical experiments never yielded more than six viable solutions, and conjectures that this is the practical maximum.¹¹

⁸The 2012 version of this note debuted the h - j - k polynomials, but had not articulated the “Mazur+Pseudoface=1” Principle nor offered bounds on tetrahedral solutions besides the vague “finitely many”. This author thanks Ya-Lun Tsai for shaming his complacency with that phrase.

⁹Rather, they gave the squares of the side-lengths, from which we derive those metrics:

$$(a^2, b^2, c^2, d^2, e^2, f^2) = (16, 25, 9, 9, 33, 54)$$

¹⁰Somewhat remarkably, each of p_j and p_k has its own full roster of nine *real* roots; the extraneous values are rejected for being merely negative. This author’s experiments have tended to generate non-real roots in one or more of the polynomials.

¹¹Could this bound have something to do with the “messy sextic”?

Whatever the *practical* maximum, we can concoct Mazur metrics that reduce the *theoretical* maximum by zeroing-out trailing terms of the h -polynomial. (Zeroing-out the leading term requires a non-scalenohedral tetrahedron.) For instance,

- With $(w, x, y, z, u) = (1, 3, 2, 7, 1)$, we find that $su^3 = \frac{26531}{25}$ causes p_h 's constant term to vanish. Ignoring a factor of h , the polynomial's effective degree—and thus the upper bound on potential tetrahedra—reduces to eight. As it happens, this particular polynomial has only four h -roots corresponding to viable tetrahedra:

$$(18) \quad (h, j, k) = \begin{cases} (2.2085\dots, 5.1311\dots, 5.6603\dots) \\ (5.2054\dots, 4.9004\dots, 2.8941\dots) \\ (7.1474\dots, 2.8580\dots, 2.9944\dots) \\ (7.2068\dots, 1.6576\dots, 4.1355\dots) \end{cases}$$

- With $(w, x, y, z) = (1, 2, 3, 4)$, we find $u^2 = 0.8313\dots$ and $su^3 = 37.7585\dots$ make both the constant and linear coefficients vanish, for a maximum positive root count of seven. This particular p_h has only two positive roots, for two viable tetrahedra:

$$(19) \quad (h, j, k) = \begin{cases} (0.2342\dots, 2.6721\dots, 7.0935\dots) \\ (1.4493\dots, 0.6327\dots, 7.9178\dots) \end{cases}$$

- With $(w, x, y) = (1, 2, 3)$, we find $z = 1.4063\dots$, $u^2 = 0.4268\dots$, $su^3 = 10.1691\dots$ make the constant, linear, and quadratic coefficients vanish, for a maximum positive root count of six. This particular p_h has three positive roots, but leads to only two viable tetrahedra:

$$(20) \quad (h, j, k) = \begin{cases} (0.3728\dots, 5.8859\dots, 1.1475\dots) \\ (2.0563\dots, 5.1990\dots, 0.1509\dots) \end{cases}$$

Alternative values $z = 4.4053\dots$, $u^2 = 0.0753\dots$, $su^3 = 3.9516\dots$ lead to three positive roots, but only one tetrahedron:

$$(21) \quad (h, j, k) = (0.5983\dots, 0.5589\dots, 9.2479\dots)$$

- With $(w, x) = (1, 2)$, it turns out that one of y and z must also be 1 or 2 to zero-out p_h 's cubic-and-lower terms; this makes the tetrahedron non-scalenohedral. Additional experiments have yielded similar results, but it's not clear that equal areas *must* appear.

The final example aside, the number of Mazur-determined scalenohedral tetrahedra seems to consistently fall short of the theoretical maximum. Perhaps more-comprehensive numerical searches can do better; or perhaps there are conjecturable practical maxima for these cases, too.

2.2. The Bisohedral Case ($W, X, Y = Z$).

It depends, Mr. Mazur. Typically, your metrics determine at most four *strictly*-bisohedral tetrahedra. Exceptionally, the tetrahedron is unique.

When at least two face-areas—here, Y and Z —are equal, the associated pseudoface polynomial p_h collapses; its “messy sextic” component vanishes, leaving the

product of h and the square of a quartic. We define p'_h as that quartic:

$$(22) \quad p'_h(w, x, y, y; u; s) := h^4 (w - y)(x - y) \\ - h^3 (u^2 - 3y(w - x)^2) \\ + h^2 (u^2 - 3y(w - x)^2) (w + x + 2y) \\ - h (su^3 + y(w - x)^2 (4(w - y)(x - y) - (w + x + 2y)^2)) \\ + (u^2 - 3y(w - x)^2) (u^2 + y(w - x)^2)$$

That there are at most four resulting tetrahedra is guaranteed by MP1; this improves Tsai's bound of eight.¹² We can also deduce that fact from the reduced sum of squares (4) and volume (8) formulas:

$$(23) \quad h + j + k = w + x + 2y \quad u^2 = hjk - h(w - y)(x - y) - y(w - x)^2$$

For a particular root $h \neq 0$, we easily solve these relations for the coefficients of the Vieta quadratic $\mu^2 - \mu(j + k) + jk$ whose roots are h 's companion j and k values.

2.2.1. *The exceptional condition.* Observe that the condition $u^2 = 3y(w - x)^2$ causes most of p'_h to vanish. Discarding a factor of h , we can express what remains as

$$(24) \quad p_h^*(\dots) := h^3(w - y)(x - y) - y(w - x)^2 (3su - 8y(w + x) - (w - x)^2)$$

For Y distinct from W and X , this has at most one positive root, an explicit *cube* root. The condition has restricted the number of *strictly-bisohedral* Mazur-determined tetrahedra to at most one, dramatically improving Tsai's bound of six.¹³ Moreover, non-degeneracy requires that the long factor of the polynomial's constant term be non-zero; to ensure a positive root, the factor must have the same sign as the leading coefficient. This provides a necessary condition on the viability of proposed Mazur metrics.

Contrariwise, taking $Y = W$ (similarly, $Y = X$), we have a clear path to zeroing-out the entire h -polynomial. The resulting tetrahedron is *trisohtedral*, a case considered in §2.4.

2.3. The Doubly-Bisohedral Case ($W = X$ and $Y = Z$).

No, Mr. Mazur. Your metrics determine up to four (but perhaps only up to three) *strictly-doubly-bisohedral* tetrahedra.

When two pairs of face-areas match —here, $W = X$ and $Y = Z$ — the corresponding pseudoface quartic (22) reduces as follows:

$$(25) \quad p'_h(x, x, y, y; u, s) = h^4(x - y)^2 - h^3u^2 + 2h^2u^2(x + y) - hsu^3 + u^4$$

This form tells us nothing of significance that its merely-bisohedral counterpart didn't, although here we have no multiple-coefficient-zeroing "exceptional case".¹⁴ (When $x = y$, the tetrahedron is *equihedral*, a case covered in §2.5.) Thus, we can say only that Mazur metrics determine up to four *strictly-doubly-bisohedral* tetrahedra.

¹²It's worth noting that p_j and p_k (identical here) have degree eight. If not for MP1 and our ability to choose pseudoface H as our Mazur-augmenting parameter, eight would be our bound.

¹³The exceptional condition reduces p_j and p_k to degree six. See previous footnote.

¹⁴For $x \neq y$, octics p_j and p_k also have non-zero leading and trailing coefficients. Also, note that the bisohedral exceptional condition does not apply here, since we do not allow u to vanish.

The Lisoněk-Israelael pair in [1] falls into this category, with $X = \frac{1}{4}\sqrt{7}$, $Y = \frac{1}{2}$, $V = \frac{1}{12}\sqrt{3}$, $r = \frac{1}{6}\sqrt{21}$ (so, $x = \frac{7}{16}$, $y = \frac{1}{4}$, $u = \frac{3}{16}$ and $s = \frac{7}{3}$). These give

$$(26) \quad p'_h(\cdots) = (4h - 1)(64h^3 - 48h^2 + 76h - 9)$$

with two real roots, $h = \frac{1}{4}$ and $h = \frac{1}{4} - \frac{1}{2}\sqrt[3]{\frac{9+\sqrt{849}}{18}} + \frac{1}{2}\sqrt[3]{\frac{-9+\sqrt{849}}{18}} = 0.1268\dots$. The simple h has companion parameters $(j, k) = (\frac{3}{16}, \frac{15}{16})$; relations (11) convert these to the squared-lengths of edges: $(a^2, b^2, c^2, d^2, e^2, f^2) = (1, 1, 2, 2, 1, 2)$. The complicated h leads to $(j, k) = (0.3462\dots, 0.9019\dots)$, and thence to $a^2 = 0.5907\dots$, $d^2 = 1.0982\dots$, $b^2 = e^2 = 1.7122\dots$, $c^2 = f^2 = 2.0881\dots$

A computer search has revealed numerous examples of Mazur metrics that determine *three* distinct tetrahedra. For instance, $(X, Y, V, r) = (1, \frac{23}{25}, \frac{33}{100}, 1)$ yields these pseudoface-area parameters:

$$(27) \quad (h, j, k) = \begin{cases} (0.3490\dots, 1.5329\dots, 1.8108\dots) \\ (1.3265\dots, 0.3756\dots, 1.9906\dots) \\ (2.2993\dots, 0.4867\dots, 0.9067\dots) \end{cases}$$

By Descartes' Rule of Signs, an even number of sign changes in the coefficients of p'_h implies an even number of positive roots; if there are *three* positive roots here, there must be *four*. However, in all observed cases where p'_h admits four positive roots, one of those roots has been too large to be viable. This author has not investigated whether this is always so.

2.4. The Trisoedral Case (W and $X = Y = Z$).

Yes and (very) No, Mr. Mazur. Typically, your metrics determine at most one *strictly*-trisoedral tetrahedron. Exceptionally, a *continuum* of them.

When three face-areas match, the bisohedral quartic (22) collapses to a trisoedral cubic:

$$(28) \quad p''_h(w, x, x, x; u; s) = h^3(u^2 - 3x(w - x)^2) - h^2(u^2 - 3x(w - x)^2)(w + 3x) + h(su^3 - x(w - x)^2(w + 3x)^2) - (u^2 - 3x(w - x)^2)(u^2 + x(w - x)^2)$$

so that, when the leading coefficient doesn't vanish, MP1 limits the number of possible Mazur-determined tetrahedra to three. That said, it's more instructive to consider directly the reduced sum of squares (4), volume (8), and radius (12) formulas, which we can express thusly:

$$(29) \quad h + j + k = w + 3x \quad hjk = u^2 + x(w - x)^2$$

$$(30) \quad (hj + jk + kh)(u^2 - 3x(w - x)^2) = su^3 - x(w - x)^2(w + 3x)^2$$

Note the appearance of the cubic's leading and linear coefficients in (30). If these expressions are non-zero, then we can solve for the coefficients of a Vieta cubic $\mu^3 - \mu^2(h + j + k) + \mu(hj + jk + kh) - hjk$, whose roots form a complete parametric set: h, j, k . Therefore, there is actually *at most one* Mazur-determined tetrahedron (up to isometry).

2.4.1. *The exceptionally-exceptional condition.* When $u^2 = 3x(w-x)^2$, the leading coefficient of p_h'' is zero; indeed, by (30), *all* coefficients are zero: the polynomial simply ceases to constrain h . Only the sum-of-squares and volume formulas (29) govern the psuedoface-areas; consequently, the possibility exists for them to define a *continuum* of tetrahedra. These would be parameterized by any two psuedoface-areas—say, h and j —on the cubic curve

$$(31) \quad hj(w + 3x - h - j) = 4x(w - x)^2$$

The Law of Opposite Cosines (5) restricts h , and likewise j , to lie between $(W - X)^2$ and $(W + X)^2$ (which, in this case, are tighter bounds than $(Y - Z)^2 = 0$ and $(Y + Z)^2 = 4X^2$). When a non-trivial arc of the cubic exists within those bounds, the Mazur metrics determine infinitely-many tetrahedra.

The Yang-Zeng cubic discussed in this note's introduction provides a specific instance of this scenario, using Mazur metrics $(W, X, V, r) = (84\sqrt{3}, 63\sqrt{3}, 441, 43\sqrt{3}/6)$ (so, $(w, x, u, s) = (21168, 11907, 194481, 1849/3)$). Equation (2) is the counterpart of (31).

2.4.2. *Viability for the continuum.* The exceptionally-exceptional condition implies

$$(32) \quad hjk = 4x(w - x)^2$$

But, then, the Arithmetic-Geometric Means Inequality tells us

$$(33) \quad (h + j + k)^3 \geq 3^3 hjk \quad \rightarrow \quad (w + 3x)^3 \geq 108x(w - x)^2$$

That is, defining $\lambda := \frac{W}{X} \neq 1$,

$$(34) \quad \left((3 + \lambda)^3 - 36(3 + \lambda) + 72 \right) \left((3 - \lambda)^3 - 36(3 - \lambda) + 72 \right) \leq 0$$

We can find where the inequality is an equality explicitly:

$$(35) \quad 3 \pm \lambda = 4\sqrt{3} \cos\left(\frac{5}{18}\pi + \frac{2}{3}n\pi\right) \quad n = 0, 1, 2$$

For distinct positive W, X subject to the Tetrahedron Inequality (7), $\lambda \leq 3$, we find that we have this range of viability for *strictly*-trisoedral tetrahedra:

$$(36) \quad \underbrace{3 - 4\sqrt{3} \sin \frac{1}{9}\pi}_{0.6304\dots} \leq \lambda < 1 \quad \text{or} \quad 1 < \lambda \leq \underbrace{-3 + 4\sqrt{3} \sin \frac{2}{9}\pi}_{1.4533\dots}$$

For face-areas satisfying the strict inequalities (as in the Yang-Zeng example, which has $\lambda = \frac{4}{3}$), the Mazur metrics

$$(37) \quad \left(\lambda^2 x, x, x, x; u^2 = 3X^3 (\lambda^2 - 1)^2; s = \frac{\sqrt{3} X (\lambda^2 + 3)^2}{9|\lambda^2 - 1|} \right)$$

have a continuum of viable solutions to (29), hence a continuum of corresponding tetrahedra. For face-areas satisfying either (non-unit) equalities, we necessarily have $h = j = k$; consequently, each extreme value of W/X , together with volume satisfying (32), determines a unique strictly-trisoedral tetrahedron in the form of a right pyramid with an equilateral-triangle base and non-equilateral lateral faces.

2.5. The Equihedral Case ($W = X = Y = Z$).

Yes, Mr. Mazur. Your metrics determine at most one equihedral tetrahedron (but you already knew this).

When all face-areas match, we may ignore the h - j - k polynomials *per se*. Relations (4), (8), (12) reduce to

$$(38) \quad h + j + k = 4w \quad hj + jk + kh = su \quad hjk = u^2$$

which are exactly the coefficients of the Vieta cubic $\mu^3 - \mu^2(h + j + k) + \mu(hj + jk + kh) - hjk$ having roots h, j, k . Consequently, the Mazur metrics determine *at most one* tetrahedron (up to isometry). Proving this known fact is one of two preliminary challenges posed by Mazur.

2.5.1. *Digression.* Mazur's second preliminary challenge is to show that face-area and radius alone determine an equihedral tetrahedron only when that tetrahedron is regular. We can meet the challenge by invoking a Newton-Maclaurin polynomial inequality thusly:

$$(39) \quad \frac{1}{3}(h + j + k) \cdot hjk \leq \frac{1}{9}(hj + jk + kh)^2 \quad \rightarrow \quad 12w \leq s^2$$

When (39) is a strict inequality, the values of h, j, k vary with dependence on u (that is, volume). On the other hand, equality forces $h = j = k$, and the result is a unique, *necessarily-regular* equihedral tetrahedron determined by face-area and radius alone.

2.6. **A Bonus Case** ($W = \frac{3}{4}\sqrt{3}r^2$). Lisoněk and Israel [1] consider another situation posed by Mazur:

Mazur's Second Question. Is every tetrahedron [that is] determined just by the areas of its faces and the radius of its circumscribed sphere [...] regular?

The authors justify an answer of "No" by noting that the case $W = \frac{3}{4}\sqrt{3}r^2$ forces face W to be an equilateral triangle inscribed in a great circle of the tetrahedron's circumsphere, already imposing sufficient structure on the figure that the remaining face-areas are enough to determine it completely. Indeed, there are only *two* degrees of freedom in locating the vertex opposite W on the circumsphere, yet *three* available parameters X, Y, Z ; therefore, we expect such a tetrahedron to be over-determined by its radius and face-areas. We provide here a (mostly-)hedronometric discussion of this configuration.

Re-notation. This section uses w, x, y, z differently than the others.

Let O be the circumcenter of the tetrahedron (and of face W), let P be the vertex opposite W , let Q be the projection of P into the plane of W . Let $w := |PQ|$ be the altitude perpendicular to W , and note that

$$(40) \quad w = \frac{2X}{d} \sin D = \frac{2Y}{e} \sin E = \frac{2Z}{f} \sin F$$

This relation holds for any tetrahedron, but here we have specifically $d = e = f = \sqrt{3}r$. Relatedly, let us define x, y, z as the *signed* distances from Q to the sides of

W , as follows:

$$(41) \quad x := \frac{2X}{d} \cos D \quad y := \frac{2Y}{e} \cos E \quad z := \frac{2Z}{f} \cos F$$

For equilateral¹⁵ W , the reader can verify this formula for the power of Q with respect to the circumcircle of W :

$$(42) \quad \text{pow}(Q) := |OQ|^2 - r^2 = -\frac{4}{3}(xy + yz + zx)$$

Importantly, the definition of $\text{pow}(Q)$ matches the Pythagorean calculation for $-|PQ|^2$ (that is, $-w^2$). Consequently, (40) and the easily-derived identity

$$(43) \quad W = X \cos D + Y \cos E + Z \cos F$$

allow us to write

$$(44) \quad \begin{aligned} w^2 &= \frac{4X^2}{d^2} \sin^2 D = \frac{4Y^2}{d^2} \sin^2 E = \frac{4Z^2}{d^2} \sin^2 F \\ &= \frac{8}{3d^2} (W^2 - X^2 \cos^2 D - Y^2 \cos^2 E - Z^2 \cos^2 F) \end{aligned}$$

from which we readily deduce

$$(45) \quad X^2 \sin^2 D = Y^2 \sin^2 E = Z^2 \sin^2 F = \frac{2}{3} (-W^2 + X^2 + Y^2 + Z^2)$$

Thus, we know the dihedral angles D, E, F , up to supplement. Since Q is confined to the circumcircle of W , at most one of D, E, F is obtuse, so that at most one of their cosines is negative. In light of (43) we see that, up to symmetry, only one assignment of signs on the non-zero terms can hold. Consequently, the shape of the tetrahedron is uniquely determined, confirming the findings of Lisoněk and Israel.

2.6.1. *A note about viability.* After three rounds of squaring, we can trade the cosines in (43) for even powers of sine, which we replace via (45) to obtain the following:

$$(46) \quad \begin{aligned} 0 &= 5\mu^4 - 2^5\mu^3 (X^2 + Y^2 + Z^2) + 2^3 \cdot 3^3 \cdot \mu^2 (X^2Y^2 + Y^2Z^2 + Z^2X^2) \\ &\quad - 2^6 \cdot 3^3 \cdot \mu X^2Y^2Z^2 + 2^4 \cdot 3^3 \cdot [XY, YZ, ZX] \end{aligned}$$

where $\mu := -W^2 + X^2 + Y^2 + Z^2$. This equation serves as a necessary condition for face-areas in this bonus case.

3. CONCLUSION

Hedronometry has proven to be especially-well-suited to the investigation of Mazur's Question(s). Rather than falling back on edge-length-based determinations of tetrahedra —introducing six auxiliary parameters to a discussion already involving six metrics— we have shown that augmenting four given face-area metrics with just three pseudoface-area parameters gets us where we need to go, and the single pseudoface polynomial (14) provides a natural, unified context for cases that had been discussed elsewhere in isolation. Moreover, the “Mazur + Pseudoface = 1” Principle has helped streamline the enumeration of solutions.

¹⁵For a not-necessarily-equilateral $\triangle ABC$, the circumcircle power of point Q with (signed) distances x, y, z from sides opposite respective vertices A, B, C is

$$\text{pow}(Q) = -\frac{yz \sin A + zx \sin B + xy \sin C}{\sin A \sin B \sin C}$$

REFERENCES

- [1] P. Lisoněk, R. B. Israel, “Metric invariants of tetrahedra via polynomial elimination”, *Proceedings of the 2000 international symposium on symbolic and algebraic computing (ISSAC '00)*. ACM, New York, NY, USA, pp 217–219. DOI=10.1145/345542.345635 <http://doi.acm.org/10.1145/345542.345635>
- [2] M. Mazur. “Problem 10717”. *The American Mathematical Monthly*, Volume 106 (1999), pg 167. <http://www.jstor.org/stable/2589061>
- [3] Y. Tsai. “On the number of tetrahedra determined by volume, circumradius and four face areas”. Preprint (2015), later published as “Estimating the number of tetrahedra determined by volume, circumradius and four face areas using Groebner basis”, *Journal of Symbolic Computation*. 77. 10.1016/j.jsc.2016.02.002.
- [4] L. Yang, Z. Zeng, “An open problem on metric invariants of tetrahedra”, *Proceedings of the 2005 international symposium on symbolic and algebraic computing (ISSAC '05)*. ACM, New York, NY, USA, pp 362–364. DOI=10.1145/1073884.1073934 <http://doi.acm.org/10.1145/1073884.1073934>