

TETRAHEDRA SHARING VOLUME, FACE AREAS, AND CIRCUMRADIUS: A HEDRONOMETRIC APPROACH

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ABSTRACT. Volume, face areas, and circumradius (here, the “Mazur metrics”) sometimes determine multiple —even infinitely-many— non-isomorphic tetrahedra. Hedronometry provides a context for unifying and streamlining previous discussions of this fact.

Marcin Mazur [2] posed the following as an open question in 1999:

Mazur’s Question. Is every tetrahedron [...] determined by its volume, the areas of its faces, and the radius of its circumscribed sphere?

We refer to volume, areas of faces, and circumradius collectively as the “Mazur metrics”.

Petr Lisoněk and Robert Israel [1] answered the Question in the negative in 2000 with the construction of two non-isomorphic tetrahedra with volume $\sqrt{3}/12$, face areas $\sqrt{7}/4$, $\sqrt{7}/4$, $1/2$, $1/2$, and circumradius $\sqrt{21}/6$; they refined Mazur’s Question to ask whether Mazur metrics determine only *finitely many* non-isomorphic tetrahedra. In 2005, Lu Yang and Zhenbing Zeng [3] exhibited a continuum of non-isomorphic tetrahedra with volume 441, face areas $84\sqrt{3}$, $63\sqrt{3}$, $63\sqrt{3}$, $63\sqrt{3}$, and circumradius $43\sqrt{3}/6$.

The Yang-Zeng tetrahedra are parameterized by points on a portion of a seemingly-contrived symmetric cubic curve in \mathbb{R}^2 . The introduction of an auxiliary variable provides an illuminating three-fold symmetry that directly relates the parameters to the areas of each tetrahedron’s three pseudo-faces, suggesting that (tetra)hedronometry — the dimensionally-enhanced analogue of trigonometry— provides a worthwhile context for consideration of Mazur’s Question.

1. PRELIMINARIES: HEDRONOMETRIC PARAMETERS

A tetrahedron is uniquely determined “trigonometrically” by the lengths (and arrangement) of its six edges; a tetrahedron is likewise determined “hedronometrically” by the areas (and arrangement) of its four faces W , X , Y , Z , and three pseudo-faces¹ H , J , K ,

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¹Geometrically: In the projection of a tetrahedron into a plane parallel to a pair of opposite edges, the quadrilateral whose diagonals are the projections of the chosen edge pair is a “pseudo-face” of the tetrahedron. Pseudo-face H is associated with the edges common to face-pairs $\{W, X\}$ and $\{Y, Z\}$; pseudo-face J , with edges common to face-pairs $\{W, Y\}$ and $\{Z, X\}$; pseudo-face K , with edges common to $\{W, Z\}$ and $\{X, Y\}$. In this note, the geometric interpretation is of no particular relevance; the reader may simply take (LoC2) as the formal definition of quantities H , J , K to which we affix the name “pseudo-face areas”.

subject to a dependence we call the Sum of Squares identity.

$$(SoS) \quad W^2 + X^2 + Y^2 + Z^2 = H^2 + J^2 + K^2$$

A tetrahedron's volume, V , is given by this hedronometric formula

$$(HV) \quad 81V^4 = 2W^2X^2Y^2 + 2W^2Y^2Z^2 + 2W^2Z^2X^2 + 2X^2Y^2Z^2 + H^2J^2K^2 \\ - H^2(W^2X^2 + Y^2Z^2) - J^2(W^2Y^2 + Z^2X^2) - K^2(W^2Z^2 + X^2Y^2)$$

and, rounding-out the Mazur metrics, its circumradius, r , is given by this formula

$$(HR) \quad 64(3V)^{10}r^2 = \left[\sqrt{[H, Y, Z][H, W, X]}, \sqrt{[J, Z, X][J, W, Y]}, \sqrt{[K, X, Y][K, W, Z]} \right]$$

where " $[\bullet]$ " is the ubiquitous "Heronic product"²

$$(1) \quad [x, y, z] := (x + y + z)(-x + y + z)(x - y + z)(x + y - z)$$

Observe that the individual Heronic products in (HR) are geometrically meaningful:

$$(2) \quad \begin{array}{lll} 9V^2 a^2 = [H, Y, Z] & 9V^2 b^2 = [J, Z, X] & 9V^2 c^2 = [K, X, Y] \\ 9V^2 d^2 = [H, W, X] & 9V^2 e^2 = [J, W, Y] & 9V^2 f^2 = [K, W, Z] \end{array}$$

for a the edge common to faces Y and Z , etc.

Note. The necessity of non-negativity for each Heronic product in (2) imposes bounds on pseudo-face areas analogous to those the Triangle Inequality imposes on edge lengths; namely,

$$(3) \quad \begin{array}{ll} |Y - Z| \leq H \leq Y + Z & |W - X| \leq H \leq W + X \\ |Z - X| \leq J \leq Z + X & |W - Y| \leq J \leq W + Y \\ |X - Y| \leq K \leq X + Y & |W - Z| \leq K \leq W + Z \end{array}$$

with equalities only in degenerate cases. These bounds are consistent with —and perhaps best-understood via— the hedronometric Second Law of Cosines:

$$(LoC2) \quad \begin{array}{lll} Y^2 + Z^2 - 2YZ \cos A & = & H^2 = W^2 + X^2 - 2WX \cos D \\ Z^2 + X^2 - 2ZX \cos B & = & J^2 = W^2 + Y^2 - 2WY \cos E \\ X^2 + Y^2 - 2XY \cos C & = & K^2 = W^2 + Z^2 - 2WZ \cos F \end{array}$$

for A the dihedral angle between faces Y and Z , etc. Finally, the *First* Law of Cosines

$$(LoC1) \quad W^2 = X^2 + Y^2 + Z^2 - 2YZ \cos A - 2ZX \cos B - 2XY \cos C$$

gives rise to the Tetrahedron Inequality

$$(TetIneq) \quad W \leq X + Y + Z$$

with equality only in degenerate cases.

²Named for Heron's classical formula, $\frac{1}{4}\sqrt{[x, y, z]}$, for the area of a triangle with side-lengths x, y, z .

2. ANSWERING MAZUR'S QUESTION

In the context of Mazur's Question, we consider face areas W, X, Y, Z to be given, so that our tetrahedra are *parameterized* by pseudo-face areas H, J, K . Ignoring trivial degeneracies³, the Sum of Squares Identity (SoS) and the hedronometric volume and circumradius formulas (HV) and (HR) constitute a system of three polynomial equations in the three parameters. Using, say, the method of resultants, we can eliminate J and K from the system to arrive at this equation

$$(4) \quad 0 = h(H, W, X, Y, Z, V, r) := H^{18} (W^2 - Y^2) (W^2 - Z^2) (X^2 - Y^2) (X^2 - Z^2) \\ + 3H^{16} (\dots) + \dots + (W^2 - X^2)^2 (Y^2 - Z^2)^2 (\dots)$$

where h is a degree-9 polynomial in H^2 . (It has degree 8 in each of W^2, X^2, Y^2, Z^2 ; degree 4 in V^4 ; and degree 2 in r^2 .) We can construct polynomials j and k for J and K by cycling $H \rightarrow J \rightarrow K \rightarrow H$ and $X \rightarrow Y \rightarrow Z \rightarrow X$.

The full nature of h, j, k is unknown to this author, but it's clear that, "usually", they provide a finite pool of values for areas H, J, K , so that a particular collection of Mazur metrics determines only finitely-many tetrahedra. Unusually, certain relations among Mazur metrics cause the polynomial's terms all to vanish, effectively providing an *infinite* pool of roots from which to draw H, J, K . We examine a few circumstances, usual and un-, revisiting the results of Lisoněk and Israel [1] and of Yang and Zeng [3] as we go.

2.1. The Equihedral Case ($W = X = Y = Z$): "Yes, Mr. Mazur (although you already knew this)". Here, as one expects, pseudo-face polynomials h, j, k coincide:

$$(5) \quad h(L, W, W, W, W, V, r) = j(L, W, W, W, W, V, r) = k(L, W, W, W, W, V, r) \\ = 6561L^2V^8 (L^6 - 4L^4W^2 + 36L^2r^2V^2 - 81V^4)^2$$

Evidently (for non-zero V, L), values H^2, J^2, K^2 are the three roots of a cubic polynomial⁴

$$(6) \quad p(M) := M^3 - 4M^2W^2 + 36Mr^2V^2 - 81V^4$$

so that, whenever V, W , and r are properties of *at least one* actual tetrahedron, they are the properties of *exactly one* tetrahedron (up to isometry). Proving this known fact is one of two preliminary challenges accompanying Mazur's general Question.

Digression. Mazur's second preliminary challenge is to show that W and r alone determine an equihedral tetrahedron only when that tetrahedron is regular. We can meet the challenge by delving into conditions under which W, r , and V produce viable tetrahedra.

³We assume throughout that the face areas, pseudo-face areas, volume, and circumradius are all non-zero, and that the dihedral angles in (LoC2) are strictly between 0 and π .

⁴We can also derive this polynomial by noting—upon reduction of each of (SoS), (HR), and (HV)—that an equihedral tetrahedron's Mazur metrics can be expressed in terms of elementary symmetric polynomials in the squares of its pseudo-face areas.

$$4W^2 = H^2 + J^2 + K^2 \quad 36r^2V^2 = H^2J^2 + J^2K^2 + K^2H^2 \quad 81V^4 = H^2J^2K^2$$

Of course, the roots of p —as squares of the pseudo-face areas— must be real; thus, the discriminant of p must be non-negative, and (for $V \neq 0$) we have this polynomial condition in $N = V^2$:

$$(7) \quad q(N) := 2187N^2 + 288Nr^2(8r^4 - 9W^2) - 256W^4(r^4 - W^2) \leq 0$$

For q —a quadratic with positive leading coefficient— to obtain non-positive values, *its* roots must both be real; thus, the discriminant of q must also be non-negative. We conclude:

$$(8) \quad 4r^4 \geq 3W^2$$

When (8) is a strict inequality, q admits a full interval of values $N = V^2$ satisfying (7); this provides a continuum of (possibly non-viable⁵) tetrahedra that share face areas and circumradius, but differ in volume, so that a complete set of Mazur metrics is required to distinguish each of them.

In the case of equality in (8), the interval of allowable N s (that is, V s) in (7) collapses to a single point — $N = V^2 = 64r^6/243$ — at which q attains the value 0; we expect, then, that polynomial p has at least two equal roots, although we find that, in fact, all three roots match — $M = H^2 = J^2 = K^2 = 16r^4/9$ — yielding a unique, viable, and necessarily-*regular* tetrahedron determined by W and r alone.

2.2. The (Strictly) Trisoedral Case ($W \neq X = Y = Z$): “No, Mr. Mazur, there can be a continuum of possibilities”. Consider now, as Yang and Zeng did in [3], the universe of tetrahedra with three faces of equal area, $X = Y = Z$; we take area W to be distinct in order to avoid the special case covered earlier. The reduced form of $h(L, W, X, X, X, V, r) = j(L, W, X, X, X, V, r) = k(L, W, X, X, X, V, r)$ reveals the values H^2, J^2, K^2 to be roots of this cubic polynomial:

$$(9) \quad p(M) := 3M^3t - 3M^2st + M \left(2916r^2V^6 - s^2X^2(W^2 - X^2)^2 \right) - 3t \left(81V^4 + X^2(W^2 - X^2)^2 \right)$$

where $s := W^2 + 3X^2$ and $t := 27V^4 - X^2(W^2 - X^2)^2$. As in the general case, we can “usually” expect to solve for uniquely-determined pseudo-face areas and infer at most one tetrahedron. Notice, however, that we can utterly *trivialize* p —making it non-determinative of H^2, J^2, K^2 — by imposing conditions on our metrics to make the polynomial’s coefficients vanish:

$$(10) \quad t = 0 \quad \text{and} \quad 2916r^2V^6 = s^2X^2(W^2 - X^2)^2$$

⁵For viability, the inequalities (3) require that the roots lie between 0 and $(2W)^2$. Applying the Descartes Rule of Signs to $p(-M)$ assures the lower bound; applying the Rule to

$$p(M + 4W^2) = M^3 + 8M^2W^2 + 4M(9r^2V^2 + 4W^4) + 9V^2(16r^2W^2 - 9V^2)$$

introduces the condition $4rW \geq 3V$ to avoid coefficient sign changes; this guarantees that the shifted roots are non-positive, whence the un-shifted roots have upper bound $4W^2$.

whence

$$(11a) \quad 27V^4 = X^2 (W^2 - X^2)^2$$

$$(11b) \quad 432r^4 = \frac{(W^2 + 3X^2)^4}{X^2 (W^2 - X^2)^2}$$

In fact, each of these conditions guarantees the other. To see this, first note the reduced hedronometric volume and circumradius formulas (HV) and (HR)

$$(12a) \quad \begin{aligned} 81V^4 &= 6W^2X^4 + 2X^6 + H^2J^2K^2 - (H^2 + J^2 + K^2)X^2(W^2 + X^2) \\ &= 6W^2X^4 + 2X^6 + H^2J^2K^2 - (W^2 + 3X^2)X^2(W^2 + X^2) \\ &= H^2J^2K^2 - X^2(W^2 - X^2)^2 \end{aligned}$$

$$(12b) \quad \begin{aligned} 2916V^6r^2 &= \left(H^2J^2K^2 - 4X^2(W^2 - X^2)^2 \right) (H^2J^2 + J^2K^2 + K^2H^2) \\ &\quad + X^2(W^2 - X^2)^2(W^2 + 3X^2)^2 \\ &= H^2J^2K^2(H^2J^2 + J^2K^2 + K^2H^2) + X^2(W^2 - X^2)^2[H, J, K] \end{aligned}$$

Together, (11a) and (12a) imply

$$(13) \quad H^2J^2K^2 = 4X^2(W^2 - X^2)^2$$

and then (11b) follows from (12b). On the other hand, squaring both sides of (12b), and then substituting r^4 from (11b), and $H^2J^2K^2 - 81V^4$ for $X^2(W^2 - X^2)^2$ (via (12a)), and $H^2 + J^2 + K^2$ for $W^2 + 3X^2$ (via (SoS)) yields

$$(14) \quad 0 = (108V^4 - H^2J^2K^2)q(V^2)$$

where q is a quadratic in V^2 with no non-zero real roots; therefore, this case requires $108V^4 = H^2J^2K^2$, which, by (12a), is equivalent to (13) and implies (11a).

As the above discussion shows, equation (13) is an unavoidable consequence (and/or catalyst) of trivializing polynomial p in (9); as such, it characterizes all non-isomorphic trisoedral tetrahedra that are not necessarily determined by their Mazur metrics. Yang and Zeng —using a different, but equivalent, parameterization⁶— effectively demonstrate that the metrics $(W, X, V, r) = (84\sqrt{3}, 63\sqrt{3}, 441, 43\sqrt{3}/6)$ are shared by a continuum of distinct trisoedral tetrahedra corresponding to a continuum of solutions to (13). This single case is enough to provide a negative answer to Mazur's Question, but we'll conclude this discussion with a digressive investigation of conditions on W and X that lead to viable generalizations of that case.

Digression. Consider what the Arithmetic-Geometric Inequality demands of H, J, K :

$$(15) \quad (H^2 + J^2 + K^2)^3 \geq 3^3 \cdot H^2J^2K^2$$

⁶Their ξ and η satisfy $H^2 = 23814(1 - \xi)$ and $J^2 = 23814(1 - \eta)$. They make no mention of an explicit counterpart for K , although we can derive one from the Sum of Squares identity.

Invoking the Sum of Squares identity on the left-hand side, and relation (13) on the right-hand side, we have

$$(16) \quad (W^2 + 3X^2)^3 \geq 108X^2 (W^2 - X^2)^2$$

Equivalently,

$$(17) \quad (W(9X^2 - W^2) + 9X(X^2 - W^2)) (W(9X^2 - W^2) - 9X(X^2 - W^2)) \geq 0$$

where we've grouped each factor to separate the always-non-negative⁷ quantity $W(9X^2 - W^2)$ from the ambiguously-signed quantity $\pm 9X(X^2 - W^2)$. When $X > W$ (respectively, $W/3 \leq X < W$), the first (second) factor of (17) is non-negative, and we need only require that the second (first) factor be as well:

$$(18) \quad \begin{aligned} X > W \quad \text{and} \quad W(9X^2 - W^2) - 9X(X^2 - W^2) &\geq 0 \\ \text{OR} \quad \frac{1}{3}W \leq X < W \quad \text{and} \quad W(9X^2 - W^2) + 9X(X^2 - W^2) &\geq 0 \end{aligned}$$

whence

$$(19) \quad \begin{aligned} 1 < \frac{X}{W} \leq \frac{1}{3} \left(1 + 4 \cos \frac{\pi}{9} \right) &\approx 1.586 \dots \\ \text{OR} \quad 0.688 \dots \approx \frac{1}{3} \left(-1 + 4 \cos \frac{2\pi}{9} \right) &\leq \frac{X}{W} < 1 \end{aligned}$$

For X/W *strictly* within the bounds of (19), we have a continuum of (H, J, K) satisfying the Sum of Squares identity and (13); hence, a continuum of (H, J) satisfying this equation

$$(20) \quad H^2 J^2 (W^2 + 3X^2 - H^2 - J^2) = 4X^2 (W^2 - X^2)^2$$

that generalizes the cubic relation of the Yang-Zeng parameters. Each solution corresponds to a trisoedral tetrahedron with volume V and circumradius r given by (11a) and (11b), amplifying Yang and Zeng's answer of "No" to Mazur's Question.

Having X/W at the non-unit endpoint of either range in (19) gives equality in (15), so that, necessarily, $H = J = K$. Consequently, each of these endpoints corresponds to a circumstance in which face areas and circumradius (or face areas and volume) determine a unique trisoedral tetrahedron with three-fold rotational symmetry about an equilateral base, and we can express the volume V , circumradius r , and pseudo-face areas in terms of the face areas W and X :

$$(21a) \quad V^4 = \frac{1}{2916} (W^2 + 3X^2)^3$$

$$(21b) \quad r^4 = \frac{1}{4} (W^2 + 3X^2)$$

$$(21c) \quad H^2 = J^2 = K^2 = \frac{1}{3} (W^2 + 3X^2)$$

⁷By the Tetrahedron Inequality, $W \leq X + Y + Z$. (TetIneq)

Note. That W , X , and r alone suffice to determine the full results of (21) provides the answer “No” to the second open question from Mr. Mazur:

Mazur’s Second Question. Is every tetrahedron [that is] determined just by the areas of its faces and the radius of its circumscribed sphere [...] regular?

2.3. The Doubly-Bisohedral Case ($W = X \neq Y = Z$): “No, Mr. Mazur, but the possibilities are finite”. Here, polynomials j and k are identical, but cumbersome:

$$(22) \quad j(L, X, X, Y, Y, V, r) = k(L, X, X, Y, Y, V, r) = p_{jk}(L^2)$$

with

$$(23) \quad p_{jk}(M) := M^8 (X^2 - Y^2)^2 + M^7 \left(81V^4 - 8(X^2 - Y^2)^2 (X^2 + Y^2) \right) \\ + \dots + (X^2 - Y^2)^4 (\dots)$$

On the other hand, h reduces rather compactly:

$$(24) \quad h(H, X, X, Y, Y, V, r) = H^2 (p_h(H^2))^2$$

with

$$(25) \quad p_h(M) := M^4 (X^2 - Y^2)^2 - 81M^3V^4 + 162M^2V^4 (X^2 + Y^2) \\ - 2916Mr^2V^6 + 6561V^8$$

Assuming non-zero X, Y, V, r , and with $X \neq Y$, we see that we cannot, as we did in the trisohedral case, trivialize p_h or p_{jk} by forcing their coefficients to simultaneously vanish; therefore, H^2 , J^2 , K^2 are constrained to be roots of these polynomials, so that at most finitely-many double-bisohedral tetrahedra are ever determined by a set of Mazur metrics.

Indeed, from the next-to-leading coefficients in monic-ized versions of (23) and (25), we see that the sum of all possible roots of these polynomials —and thus, the sum of all possible H^2 , J^2 , and K^2 — is

$$(26) \quad \frac{-81V^4 + 8(X^2 - Y^2)^2 (X^2 + Y^2) + 81V^4}{(X^2 - Y^2)^2} = 8(X^2 + Y^2) \\ = 4(W^2 + X^2 + Y^2 + Z^2)$$

suggesting that those twelve roots partition into exactly four H^2 - J^2 - K^2 triads, each of which satisfies the Sum of Squares identity and determines a tetrahedron. We have not ruled-out the possibility of each available H^2 combining with roots of (25) in more than one way, so there may be more than four tetrahedral solutions available.

The Lisoněk-Israel example in [1] has $(X, Y, V, r) = (\sqrt{7}/4, 1/2, \sqrt{3}/12, \sqrt{21}/6)$, whence

$$(27a) \quad p_h(M) \sim (4M - 1) (64M^3 - 48M^2 + 76M - 9)$$

$$(27b) \quad p_{jk}(M) \sim (16M - 3) (16M - 15) \begin{pmatrix} 16777216M^6 - 56623104M^5 \\ + 87228416M^4 - 60571648M^3 \\ + 15095552M^2 - 901728M + 15957 \end{pmatrix}$$

so that (filtering-out non-real roots)

$$(28) \quad H^2 \in \left\{ \frac{1}{4}, 0.1268\dots \right\} \quad J^2, K^2 \in \left\{ \frac{3}{16}, \frac{15}{16}, 0.3462\dots, 0.9019\dots \right\}$$

subject to the Sum of Squares constraint

$$(29) \quad H^2 + J^2 + K^2 = W^2 + X^2 + Y^2 + Z^2 = \frac{11}{8}$$

Without loss of generality, we take $J \leq K$, and the two determined tetrahedra correspond to these pseudo-face parameters:

$$(30) \quad (H^2, J^2, K^2) = \left(\frac{1}{4}, \frac{3}{16}, \frac{15}{16} \right) \quad \text{or} \quad (0.1268\dots, 0.3462\dots, 0.9019\dots)$$

A computer search reveals numerous examples of Mazur metrics that determine three doubly-bisohedral tetrahedra. For example, $(X, Y, V, r) = (1, \frac{23}{25}, \frac{33}{100}, 1)$ determines tetrahedra with these parameters:

$$(31) \quad (H^2, J^2, K^2) = (0.3490\dots, 1.5329\dots, 1.8108\dots) \\ \text{or} \quad (1.3265\dots, 0.3756\dots, 1.9906\dots) \\ \text{or} \quad (2.2993\dots, 0.4867\dots, 0.9067\dots)$$

No metrics giving rise to four tetrahedra have been found. In all observed cases where p_h admits four non-negative real roots, one of those roots is (sometimes a great many) orders of magnitude too large. We have not investigated whether this is always so.

2.4. A Bonus Case: $W = \frac{3\sqrt{3}}{4}r^2$. Lisoněk and Israel [1] justify their own answer of "No" to Mazur's Second Question⁸ by observing that the condition, say, $W = \frac{3\sqrt{3}}{4}r^2$ forces face W to be an equilateral triangle in an equatorial plane of the tetrahedron's circumsphere; they deduce that such a (not-necessarily-regular) tetrahedron is uniquely determined by its circumradius and face areas. We can approach their conclusion hedronometrically, but without the h, j, k polynomials from (4).

We begin with this hedronometric fact for which we lack hedronometric proof:⁹

Fact. *Given a tetrahedron with an equilateral triangle "base" face inscribed in a great circle of its circumsphere, the sum of the base area and any other face area is greater than the sum of the remaining face areas:*

$$(32) \quad W + X \geq Y + Z \quad W + Y \geq Z + X \quad W + Z \geq X + Y$$

with equality only in the degenerate case of coplanar faces.

⁸See the very end of Section 2.2.

⁹We resort to coordinates, situating the vertices of face W at $(r, 0, 0)$ and $(-\frac{1}{2}r, \pm\frac{1}{2}r\sqrt{3}, 0)$, and the fourth vertex at $(p, q, \sqrt{r^2 - p^2 - q^2})$. The areas of the faces in terms of p, q, r are given by these equations:

$$16W^2 = 9r^2 \quad 16X^2 = 5r^2 + 4rp - 4q^2 \quad \{16Y^2, 16Z^2\} = 5r^2 - 2rp - 3p^2 - q^2 \pm 2q(r - p)\sqrt{3}$$

Eliminating face areas from the equation $W + X = Y + Z$ yields the relation $p^2 + q^2 = r^2$. As $W + X > Y + Z$ for $(p, q) = (0, 0)$ the result follows, by continuity.

Now, the bottom three equations in (2) —with $d = e = f$ — and the Sum of Squares identity (SoS) provide a system of four polynomial equations in parameters H, J, K, Vd . Eliminating the last three parameters leaves a polynomial equation in H^2 that simplifies under the substitution $H^2 = W^2 + X^2 - 2WX \cos D$; writing M for $X \cos D$, we have:

$$(33) \quad 0 = p(M) := 3M^4 - 4M^3W - 2M^2(W^2 + 2X^2 - Y^2 - Z^2) \\ + 4MW(W^2 + 2X^2 - Y^2 - Z^2) + [W, Y, Z] - 4W^2X^2$$

Consider the behavior of p at key points, with the Tetrahedron Inequality (TetIneq) and the Fact allowing us to determine the sign of the first two expressions (specifically for non-degenerate cases):

$$(34a) \quad p(-X) = -(W+X-Y-Z)(W+X+Y-Z)(W+X-Y+Z)(W+X+Y+Z) < 0$$

$$(34b) \quad p(X) = (-W+X+Y+Z)(W-X+Y-Z)(W-X-Y+Z)(W-X+Y+Z) > 0$$

$$(34c) \quad p(W) = -(Y^2 - Z^2)^2 \leq 0$$

We see, then, that p has a root such that $M < -X$; another root in the range $X < M \leq W$;¹⁰ and a third with $W \leq M$. (These last two coincide as a double-root when $Y = Z$, as $M = W$ determines a critical point —a relative minimum— of p .) Since $-X < X \cos D < X$ in our non-degenerate tetrahedron, these roots are extraneous, leaving at most one viable value for $\cos D$ and hence H ; likewise, J and K . The tetrahedron is therefore uniquely determined by its face areas and circumradius, as claimed.

3. CONCLUSION

Hedronometry has proven to be especially-well-suited to the investigation of Mazur's Question(s). Rather than falling back on edge-length-based determinations of tetrahedra —introducing six auxiliary parameters to a discussion already involving six metrics— we have found that augmenting given face area metrics with just three pseudo-face area parameters gets us where we need to go, and the single pseudo-face polynomial (4) provides a natural, unified context for the cases that had been discussed before.

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¹⁰The tetrahedron has at most one equilateral face, so $X < W$.