THE DESCARTES RULE OF SWEEPS AND THE DESCARTES SIGNATURE

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Abstract. The Descartes Rule of Signs, which establishes a bound on the number of positive roots of a polynomial with real coefficients, is extended to polynomials with complex coefficients. The extension is modified to bound the number of complex roots in a given direction on the complex plane, giving rise to the Descartes Signature of a polynomial.

The search for the roots of a polynomial is sometimes aided by the following result, often taught in high school:

Theorem 1 (The Descartes Rule of Signs)**.** *Let*

$$p(x) := a_0 x^{m_0} + a_1 x^{m_1} + a_2 x^{m_2} + \cdots + a_n x^{m_n}$$

be a polynomial with real coefficients $a_0, a_1, \ldots, a_n$, all non-zero, and integer exponents $0 \leq m_0 < m_1 < \cdots < m_n$.[1] *The number of positive roots of p is then at most the number of sign changes in the coefficient sequence. More specifically, the number of positive roots differs from the number of sign changes by an even number.*

Discussion and proof of the Rule of Signs can be found in the mathematical literature dating back to Descartes' own work in 1637, as well as online.[2]

As fascinating and elegant as the Rule may be, it never seemed entirely *satisfying*. One learns early on in mathematics that even the study of quadratic polynomials isn't "complete" without consideration of non-real complex numbers, yet this iconic element of polynomial lore all but ignores them.[3]

As we show, making up for this oversight requires only re-thinking sign changes in a manner that ties in quite nicely with the conceptually-illuminating revelation that "multiplication by a negative" in arithmetic corresponds to a geometric half-turn about the origin of the complex plane. Unfortunately, our proof of the adapted Rule of Signs relies on Descartes' original. Therefore, while the rotational view of complex multiplication makes the arithmetic intuitive, our new interpretation of the Rule of Signs does not (yet) provide any "ah-ha!" insights into the result.

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[1] In our notation, the traditional (lowest-power) "trailing term" comes first and (highest-power) "leading term" comes last; we will refer to them as the "initial term" and "final term", respectively.

[2] See, for instance, http://www.cut-the-knot.org/fta/ROS2.shtml

[3] Of course, using the Rule, one can occasionally glean *some* information about the number of non-real roots: subtract the maximum number of positive and (after a standard trick) negative roots from the polynomial's degree.

1. The Rule of Sweeps

1.1. Preliminaries. Let

$$r(x) := c_0 x^{m_0} + c_1 x^{m_1} + \cdots + c_n x^{m_n} \tag{1}$$

be a polynomial with complex coefficients, all non-zero, and integer exponents $0 \leq m_0 < m_1 < \cdots < m_n$. We can write $r(x) = p(x) + i \cdot q(x)$, where

$$p(x) := a_0 x^{m_0} + a_1 x^{m_1} + \cdots + a_n x^{m_n} \qquad q(x) := b_0 x^{m_0} + b_1 x^{m_1} + \cdots + b_n x^{m_n} \tag{2}$$

and $c_k = a_k + i b_k$ with real a_k and b_k for each k. Clearly, any positive root of r is a positive root of both p and q, so our analysis of r amounts to a tandem analysis of p and q via the Rule of Signs, with one provision: although the coefficient sequence $\{c_k\}$ is not all-zero, (at most) one of the sequences $\{a_k\}$ or $\{b_k\}$ might be, defying attempts to count sign changes; we shall therefore agree that

Agreement. *The number of sign changes in an all-zero coefficient sequence is infinite.*

As any number whatsoever is a root of an identically-zero polynomial, this Agreement allows us to preserve the conclusion that the number of roots is no greater than the number of coefficient sign changes, and we may write

$$\text{\# of positive roots of } r \leq \min\left(\text{\# of sign changes in } \{a_k\}, \text{\# of sign changes in } \{b_k\}\right) \tag{3}$$

Our strategy is to compute an upper bound on right-hand side of (3), using the coefficient sequence $\{c_k\}$. To do this, we introduce *sweeps*.

1.2. Sweeps defined. Imagine a needle with one end anchored at the origin of the complex plane and with the other end initially pointing in the direction of c_0. Let the free end of the needle *sweep* —always counter-clockwise— to point in the directions of c_1, c_2, etc., and finally c_n, "stalling in place" when successive coefficients are identical, and tracing out a "sweep spiral". (See Figure 1.)

We define the *positive sweep of the polynomial* as the total angle swept by the counter-clockwise needle, computing it thusly:[4]

$$sweep^+(r) := \sum_{k=1}^{n} arg^+(c_k/c_{k-1}) \qquad \text{where } 0 \leq arg^+(z) < 2\pi \tag{4}$$

Likewise, we define the *negative sweep of the polynomial* as (the absolute value of) the total angle swept by the needle if it were to move always clockwise. This is equivalent to a counter-clockwise sweep with the needle taking the coefficients in reverse order:

$$sweep^-(r) := \sum_{k=1}^{n} arg^+(c_{k-1}/c_k) \tag{5}$$

[4]Equivalently (and perhaps more in keeping with the spiral concept), we associate with the coefficient sequence $\{c_k\}$ a non-decreasing "argument sequence" $\{\theta_k\}$, where $\theta_0 = arg(c_0)$ and, for $0 < k \leq n$, each θ_k is the smallest value such that $\theta_k = arg(c_k) \pmod{2\pi}$ and $\theta_{k-1} \leq \theta_k$. Each θ_k serves as a kind of mile marker, indicating the position of c_k along the sweep spiral. The positive sweep, as the total angular distance, is then $\theta_n - \theta_0$. The negative sweep arises from a similarly-constructed *non-increasing* argument sequence.

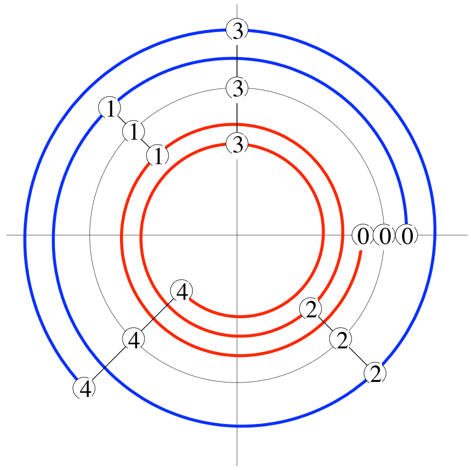


FIGURE 1. Sweep spirals for the polynomial $r(x) := 1 + (-1+i)x + (1-i)x^2 + ix^3 + (-1-i)x^4$. Dots labeled "$k$" correspond to the coefficient of x^k. The (blue) outer spiral and (red) inner spiral indicate $sweep^+(r) = 19\pi/4$ and $sweep^-(r) = 13\pi/4$.

As Figure 1 suggests, differently-directed sweeps are usually not equal. The smaller value will optimize our results, so we define *the* (undirected) sweep accordingly:

$$sweep(r) \quad := \quad \min(\ sweep^+(r),\ sweep^-(r)\ ) \tag{6}$$

This minimal sweep is guaranteed not to exceed $n\pi$: when $\arg c_k \neq \arg c_{k-1}$, we have $arg^+(c_k/c_{k-1}) + arg^+(c_{k-1}/c_k) = 2\pi$; when $\arg c_k = \arg c_{k-1}$, that sum is 0. Thus, $sweep^+(r) + sweep^-(r) \leq 2n\pi$, so that at most one sweep (and, therefore, their minimum) is less than or equal to $n\pi$.

To be slightly more precise (and to facilitate discussion is Section 2.1), we define a *staller* as a coefficient c_k (with $0 < k \leq n$) such that $\arg c_k = \arg c_{k-1}$. (That is, each staller represents an instance of the sweeping needle stalling in place.) Then,

$$sweep^+(r) + sweep^-(r) = 2(n-s)\pi \qquad \text{where} \quad s := \text{the number of stallers of } r \tag{7}$$

Consequently,

$$sweep(r) \leq (n-s)\pi \tag{8}$$

1.3. **The New Rule.** Note that a polynomial's entire family of roots (positive, negative, zero, or even non-real) is preserved if we multiply through by any non-zero complex constant. Note also that the sweeps are preserved: arithmetically, the multiplier cancels in the quotient within each term of the underlying computational definitions (4) and (5); geometrically, as points in the complex plane, the resulting polynomial's coefficients —and the sweep spirals joining them— are obtained from the original polynomial's coefficients —and sweep spirals— via rotation about the origin through a common angle (namely, the argument of the multiplied value).[5] We may therefore make the simplifying assumption that the polynomial's initial coefficient is a positive real number, so that the sweeping needle starts out directed along the positive ray of the x axis. In this scenario, upper bounds on the number of sign changes in $\{a_k\}$ and $\{b_k\}$ (from (2)) can be computed directly from the sweeps.

[5]The coefficients are also dilated in the origin by a common positive factor (the modulus of that multiplied value), although this is irrelevant to sweep computations as sweeps depend only upon direction.

The number of sign changes in $\{a_k\}$ is at most the number of times the sweeping needle *crosses* —not merely *meets*— the y axis:

$$\# \text{ of sign changes in } \{a_k\} \;\leq\; \frac{1}{2}\left\lceil \frac{2}{\pi}\, sweep(r) \right\rceil \tag{9}$$

Here, the ceiling expression converts $sweep(r)$ into the number of quadrants entered by the sweep; division by 2 takes into account that only every other quadrant entry marks a y-axis crossing.

On the other hand, the sequence $\{b_k\}$ is either all-zero (with, by our Agreement, "infinitely many" sign changes) or else the number of sign changes is bounded by the number of times the needle crosses the x axis:

$$\# \text{ of sign changes in } \{b_k\} \;=\infty \;\text{ or }\; \# \text{ of sign changes in } \{b_k\} \;\leq\; \frac{1}{2}\left(\left\lceil \frac{2}{\pi}\, sweep(r) \right\rceil - 1 \right) \tag{10}$$

The minimal bound on the changes in sign (and thus on the number of positive roots) is therefore the $\{a_k\}$ bound (9) when $\{b_k\}$ is all-zero, and the finite $\{b_k\}$ bound (from (10)) otherwise. We can simplify the corresponding bounds as follows:

- When the $\{b_k\}$ are all zero, the simplified (with c_0 real) polynomial's coefficients lie on the x axis, so that $sweep(r)$ is necessarily an integer multiple of π. The $\{a_k\}$ bound reduces to $\frac{1}{\pi}sweep(r)$.

  For a non-simplified polynomial (with c_0 possibly non-real), the condition for this case is that all coefficients lie on a common line through the origin; that is, $\arg c_0 \equiv \arg c_k \pmod{\pi}$ for all k.

- In the other case, since we are bounding *integers*, we can apply the floor function to the formula to get $\left\lfloor \frac{1}{2}\left\lceil \frac{2}{\pi}sweep(r)\right\rceil - 1\right\rfloor$, which is equivalent to $\left\lceil \frac{1}{\pi}sweep(r)\right\rceil - 1$. Being somewhat averse to taking the ceiling of an expression only to subtract 1 from the result (obtaining, in most cases, the floor of the original expression), we express $\{b_k\}$ bound as "$\lfloor \frac{1}{\pi}sweep(r)\rfloor$, except 1 less when $sweep(r)$ is a multiple of π". (Note that $sweep(r)$ is a multiple of π if and only if $\arg c_0 \equiv \arg c_n \pmod{\pi}$.)

The formula $\lfloor \frac{1}{\pi}sweep(r)\rfloor$ applies to the given $\{a_k\}$ bound. Therefore, we can combine our cases by appropriately adjusting the exception in the $\{b_k\}$ bound to obtain our sweeping generalization of the Rule of Signs:

Theorem 2 (The Descartes Rule of Sweeps)**.** *Let*

$$r(x) := c_0x^{m_0} + c_1x^{m_1} + \cdots + c_nx^{m_n} \tag{11}$$

be a polynomial with complex coefficients $c_0, c_1, \ldots, c_n$*, all non-zero, and integer exponents* $0 \leq m_0 < m_1 < \cdots < m_n$*. Then,*

$$\textit{\# of positive roots of } r \;\leq\; \begin{cases} \dfrac{1}{\pi}sweep(r) - 1, & \text{if } \arg c_0 \equiv \arg c_n \not\equiv \arg c_k \pmod{\pi} \text{ for some } k \\[2ex] \left\lfloor \dfrac{1}{\pi}sweep(r) \right\rfloor, & \text{otherwise} \end{cases} \tag{12}$$

We refer to $\lfloor \frac{1}{\pi}sweep(r)\rfloor$ *as the* primary bound *and* $\frac{1}{\pi}sweep(r) - 1$ *as the* exceptional bound.

Given that we customarily focus on *monic* ($c_n = 1$) polynomials, this restatement of the Rule may be worthwhile.

Corollary 3 (The Descartes Rule of Sweeps for Monic Polynomials)**.** *A monic polynomial, r, with (non-zero) complex coefficients, has at most $\lfloor \frac{1}{\pi} sweep(r) \rfloor$ positive roots. If r's trailing coefficient is real but some other coefficient is non-real, the bound reduces by* 1*.*

Some notes:

- If r has s stallers (see (8)), we have the immediate corollary that the number of positive roots of r is at most $n-s$, or, when the exceptional bound applies, at most $n-s-1$. (Bear in mind that n is one less than the number of terms in the polynomial, not necessarily the degree of the polynomial, or even the degree after factoring out a common power of x.)

- The primary bound of the Rule of Sweeps restates (the bound from) the Descartes Rule of Signs, because each sign change counted by the Rule of Signs contributes π to a polynomial's sweep.

  Whereas the Rule of Signs specifies that the difference between the actual root count and the computed bound is always even (possibly zero), the Rule of Sweeps as given and proven here offers no such guarantee[6]; the Rule of Sweeps cannot simply inherit that aspect from the Rule of Signs (at least from our proof): while the positive roots of p (or q) may be counted in this way, the positive roots *that p and q hold in common* need not be.

  Also, while one can construct a polynomial that has the maximum number of roots allowed by the Rule of Signs for its sign-change sequence[7], Section 2.4 of this paper shows that the Rule of Sweeps can provide upper bounds that are impossible to attain.

Why the Rule of Sweeps works isn't intuitively clear. Until a proof arises that doesn't rely on the also-not-intuitively-clear Rule of Signs, this will remain something of a mystery. That said, the notion of the sweep can be seen to have some relevance to the investigation of polynomial roots: a (monic) polynomial with non-zero roots $r_1, r_2, \ldots, r_n$ has the form

$$(x - r_1)(x - r_2) \cdots (x - r_n) = x^n - (r_1 + r_2 + \cdots + r_n)x^{n-1} + \cdots + (-1)^n r_1 r_2 \cdots r_n$$

Barring reduction modulo 2π, the argument of the constant term is $\pi n + \arg r_1 + \arg r_2 + \cdots + \arg r_n$, with every positive root (via its associated "-1" multiplier) contributing π to that value; reducing modulo 2π introduces a pesky ambiguity. Yet, under that same ambiguity, that value is equal to the polynomial's coefficient sweep. *Somehow*, the Rule of Sweeps seems to be teasing out the lost nature of the constant term's unreduced argument, setting an upper bound on the number of multiples of π (hence, the number of positive roots) involved.

2. Roots in a Given Direction: The Descartes Signature

The Descartes Rule of Signs (and, now, the Rule of Sweeps) bounds the number of *positive* roots — that is, roots in (angular) direction 0— of a polynomial p. A standard strategy allows us to bound

[6]One could suggest that a polynomial's sweep, as with many other angular measures, has a built-in ambiguity, with a value only known up to some multiple of 2π. Intriguingly, variation of 2π in the sweep corresponds exactly to variation of 2 in the root count bound. Perhaps further study will show that this connection is (or is not) more than a coincidence.

[7]Schmitt, Michael. "New designs for the Descartes rule of signs". *American Mathematical Monthly*, Vol. 11, No. 2 (2004):159-164.

the number of *negative* roots —roots in direction π— as well: apply the Rule to the auxiliary polynomial, $q(x) := p(-x)$. (The positive roots of q are the negatives of the negative roots of p.) With only the Rule of Signs at hand, such root-counting analysis effectively ends once we have looked in both directions along the real axis.

The Rule of Sweeps, however, allows us to adapt standard strategy to investigate roots *in any direction on the complex plane*. While we can consider individual directions in isolation —say, seeking the number of purely-imaginary roots by looking in directions $\pi/2$ and $-\pi/2$— we will find it instructive to look in all directions at once.

Given a polynomial r as in (11), one can bound the number of roots in direction θ —that is, roots of the form $d\exp(i\theta)$ with d positive— by applying the Rule of Sweeps to an auxiliary polynomial, r_θ, defined as follows:

$$\begin{aligned} r_\theta(x) \;:=\; e^{-i\theta m_0}\, r(xe^{i\theta}) \;&=\; e^{-i\theta m_0}\left(c_0 e^{i\theta m_0} x^{m_0} + c_1 e^{i\theta m_1} x^{m_1} + \cdots + c_n e^{i\theta m_n} x^{m_n}\right) \\ &=\; c_0 x^{m_0} + c_1 e^{i\theta(m_1 - m_0)} x^{m_1} + \cdots + c_n e^{i\theta(m_n - m_0)} x^{m_n} \end{aligned}$$

If d is a positive root of r_θ, then $0 = r_\theta(d) = r(d\exp(i\theta))$, so that $d\exp(i\theta)$ is a root of r.

Note that multiplying through by $\exp(-i\theta m_0)$ keeps the initial coefficient fixed and makes the θ multipliers negative integers. Consequently, for a given θ, the coefficients of r_θ are obtained from the coefficients of r via a counter-clockwise "fanning out" (that is, with successive coefficients being rotated about the origin of the complex plane through increasing multiples of the angle θ). Figure 2 depicts the r_θ sweep spirals for θ (at or near) a multiple of $\pi/2$.

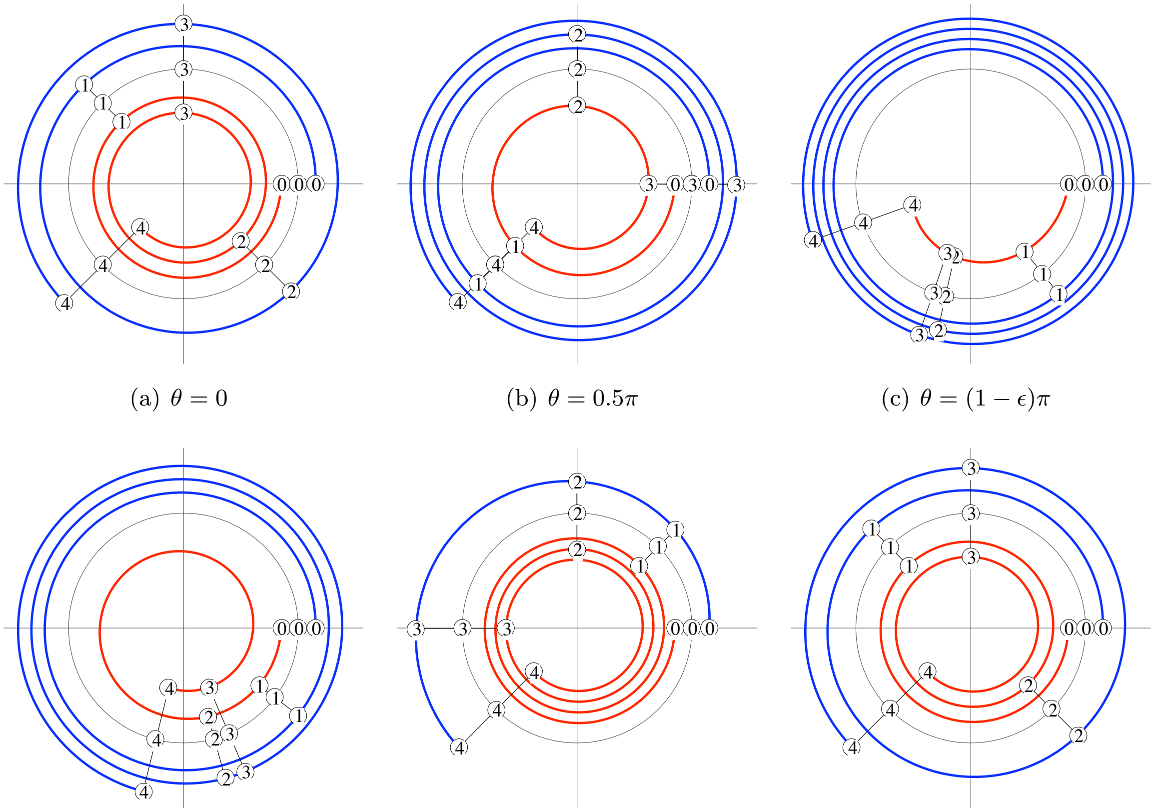
2.1. **The functions** sweep_r^+ **and** sweep_r^-**.** The fanning process, and its effect on a polynomial's directional sweeps and root count bounds, is best appreciated by considering θ as a parameter *continuously increasing* from 0 to 2π. Accordingly, we introduce notation that promotes θ to the status of function argument, and relegates r to a referential subscript:

$$\mathit{sweep}_r^+(\theta) := \mathit{sweep}^+(r_\theta) \qquad \mathit{sweep}_r^-(\theta) := \mathit{sweep}^-(r_\theta) \qquad \mathit{sweep}_r(\theta) := \mathit{sweep}(r_\theta) \tag{13}$$

Figure 3 provides a comprehensive view of the phenomenon hinted at in Figure 2, revealing the functions sweep_r^+ and sweep_r^- to be steadily increasing (or decreasing), save for a few jump discontinuities.

Various properties of the functions are fairly straightforward to derive, and these lead to explicit formulas. Throughout the following, we take r defined as in (11).

- Each graph is piecewise linear, with each piece having the same slope, namely $\pm(m_n - m_0)$ ("+" for the positive sweep, "−" for the negative sweep). To see why, ignore "wrap-around" issues and focus on the general sense that the sweep^+ (or sweep^-) measures the counter-clockwise (or clockwise) *angular distance* between the polynomial's initial and final coefficients; the auxiliary polynomial r_θ shares its initial coefficient with r, and its final coefficient arises by rotating r's final coefficient counter-clockwise by $\theta(m_n - m_0)$. Thus, the angular distance increases (decreases) at $(m_n - m_0)$-times the rate of θ.



(a) $\theta = 0$ (b) $\theta = 0.5\pi$ (c) $\theta = (1-\epsilon)\pi$

(d) $\theta = (1+\epsilon)\pi$ (e) $\theta = 1.5\pi$ (f) $\theta = 2.0\pi$

FIGURE 2. Sweep spirals for r_θ, for θ (at or near) a multiple of $\pi/2$, indicating a dramatic jump at $\theta = \pi$.

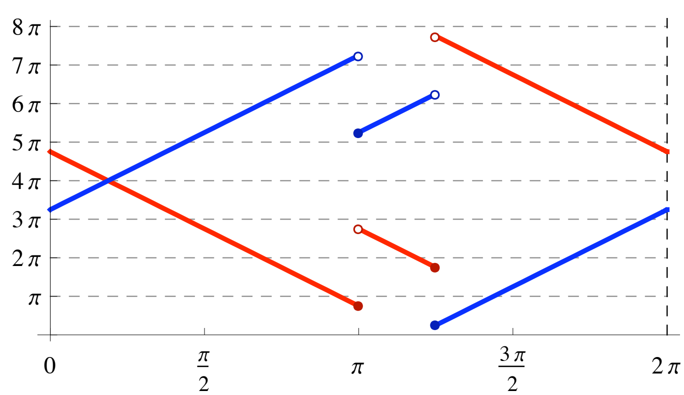


FIGURE 3. Graphs of $sweep_r^+(\theta)$ (in blue) and $sweep_r^-(\theta)$ (in red).

- The discontinuities in the graphs appear at values of θ that cause a fanned-out coefficient $c_k \exp(i\theta(m_k - m_0))$ to "catch up with" the fanned-out coefficient $c_{k-1}\exp(i\theta(m_{k-1}-m_0))$, becoming a staller for r_θ. (See Figure 2(c) and (d), indicating a discontinuity at $\theta = \pi$; another occurs at $\theta = 5\pi/4$.) Specifically, a discontinuity occurs when

$$\arg c_k + \theta(m_k - m_0) = 2m\pi + \arg c_{k-1} + \theta(m_{k-1} - m_0)$$

for some integer m, so that the multi-set of values of θ at which the sweep functions are discontinuous is

$$S_r := \bigcup_{k=1}^{n} \left\{ \frac{2m\pi - (\arg c_k - \arg c_{k-1})}{m_k - m_{k-1}} \pmod{2\pi} \,\middle|\, m = 0, 1, \dots, m_k - m_{k-1} - 1 \right\} \tag{14}$$

Defining S_r as a multi-set offers some convenient bookkeeping: the number of copies of a given θ in S_r equals the number of stallers in r_θ. Also, $|S_r| = \sum(m_k - m_{k-1}) = m_n - m_0$.

For interval I, define $\sigma_r I := |S_r \cap I|$. In particular, $\sigma_r[\theta]$ counts the stallers of r_θ.

- The vertical jump at a discontinuity is $2\pi\sigma_r[\theta]$. This is because, as a coefficient approaches its predecessor, the angular distance between them approaches 2π; when the coefficients are brought into coincidence (with one becoming a staller), the angular distance drops to 0.

  Note that the positive sweep *jumps down* at these discontinuities, so that the graph includes the point at the bottom of the jump, leaving a hole at the top. By the same token, the negative sweep *also* jumps down *when a discontinuity is approached in the opposite direction*, so that the negative sweep's graph, too, includes the point at the bottom of the jump, leaving a hole at the top. In both cases, then, a hole is positioned $2\pi\sigma_r[\theta]$ units above a point on the graph.

- As one expects, the functions are periodic, with period 2π. As θ ranges over the interval $[0, 2\pi)$, $sweep_r^+$ accumulates $2\pi(m_n - m_0)$ in value from its linear slope, but jumps downward a total of $2\pi\sigma_r[0, 2\pi) = 2\pi\,|S_r| = 2\pi(m_n - m_0)$ via the discontinuities, so that the function values at $\theta = 0$ and $\theta = 2\pi$ match. A similar argument holds for $sweep_r^-$.

- Except for the placement of holes at discontinuities, the graphs of $sweep_r^+$ and $sweep_r^-$ are mutual reflections in the horizontal line $y = n\pi$: the graph is continuous for values of θ such that r_θ has no stallers; by Equation (7) —reading "s" as $\sigma_r[\theta]$ $(= 0)$— the sum of the signed sweeps in these instances is $2n\pi$, making their average $n\pi$.

  That a hole in the graph of $sweep_r^+$ is mirrored by a point on $sweep_r^-$ (and vice-versa) follows from a simple continuity argument. Alternatively, we can note that (by the preceding bullet point) such a hole has y-value $sweep_r^+(\theta) + 2\pi\sigma_r[\theta]$, and the point has y-value $sweep_r^-(\theta)$; the hole's gain of $2\pi\sigma_r[\theta]$ in value offsets the deficit of $2\pi\sigma_r[\theta]$ in (7).

Given all of the above, we can write explicit formulas for the directional sweep functions:

$$sweep_r^+(\theta) = sweep^+(r) + \theta\,(m_n - m_0) - 2\pi\sigma_r(0, \theta] \tag{15}$$

$$sweep_r^-(\theta) = sweep^-(r) - \theta\,(m_n - m_0) + 2\pi\sigma_r[0, \theta) \tag{16}$$

Of course, because $sweep_r^+(\theta) + sweep_r^-(\theta) = 2\pi(n - \sigma_r[\theta])$, we have no need of the formula for $sweep_r^-(\theta)$. As for the (minimal) sweep function, our options remain to take $sweep_r^+(\theta)$ when the value is less than $2\pi(n - \sigma_r[\theta])$, or else $2\pi(n - \sigma_r[\theta]) - sweep_r^+(\theta)$; the explicit formulas provide us little more than the ability to separate θ-dependent computations from constants in the decision process, as the condition

$$sweep_r^+(\theta) \le sweep_r^-(\theta)$$

is equivalent to the (somewhat inelegant) condition

$$\theta(m_n - m_0) + \pi\,(\sigma_r[\theta] - 2\sigma_r(0, \theta]) \le n\pi - sweep^+(r)$$

2.2. **The Descartes Signature of a Polynomial.** Applying the Descartes Rule of Sweeps formula to the functions $sweep_r^+$ and $sweep_r^-$ —more specifically, to their minimum— provides a function that bounds the number of roots of r in each direction θ. We give this new function a special name.

Definition 1. *The Descartes Signature,* dsig*, for polynomial r (as in (11)) is the function defined by*

(17)
$$\operatorname{dsig}(r) : \theta \in [0, 2\pi) \to \begin{cases} \dfrac{1}{\pi} sweep_r(\theta) - 1, & \arg c_0 \equiv \theta(m_n - m_0) + \arg c_n \pmod{\pi}, \text{ and} \\ & \arg c_0 \not\equiv \theta(m_k - m_0) + \arg c_k \pmod{\pi} \text{ for some } k \\ \left\lfloor \dfrac{1}{\pi} sweep_r(\theta) \right\rfloor, & \text{otherwise} \end{cases}$$

We refer to $\left\lfloor \frac{1}{\pi} sweep_r(\theta) \right\rfloor$ *as the* primary component *and* $\frac{1}{\pi} sweep_r(\theta) - 1$ *as the* exceptional component *of the function.*

The graph of the Descartes signature of our on-going numerical example polynomial r appears in Figure 4. While it allows for the possibility of roots in almost any direction, it shows that at most three roots can share any given direction, and it identifies two small angular intervals (one ending at $\theta = \pi$ and another beginning at $\theta = 5\pi/4$) that serve as the direction of *no* roots. As with the original Rule of Signs, the theory of Descartes signatures is vague in general, but it may serve specific needs.

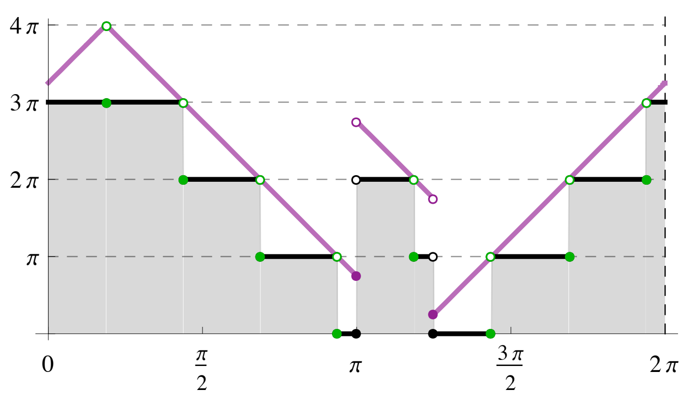


FIGURE 4. The graphs of $sweep_r$ (purple) and the (scaled-up) corresponding Descartes Signature (black and green) for the polynomial r from the preceding sub-section. (Signature values have been multiplied by π to better show the relationship with the sweep graph. Green dots indicate values at which the exceptional component of dsig is used.)

Evidently, the discontinuities in $sweep_r$ are inherited by $\operatorname{dsig}(r)$. The application of the Rule of Sweeps formula introduces additional discontinuities in $\operatorname{dsig}(r)$ at values of θ for which $sweep_r(\theta)$ is a multiple of π; but, a polynomial's sweep is, up to a multiple of 2π, the difference of its initial and final arguments, $\arg c_0 - \arg c_n$, so that these discontinuities occur when θ is a member of the set

$$T_r := \left\{ \frac{\pi m - (\arg c_n - \arg c_0)}{m_n - m_0} \pmod{2\pi} \,\middle|\, m = 0, 1, \ldots, 2(m_n - m_0) - 1 \right\} \tag{18}$$

Being in T_r puts θ halfway to satisfying the condition for the exceptional component of the definition of dsig. If we define

$$U_r := \bigcap_{k=1}^{n-1} \left\{ \frac{\pi m - (\arg c_k - \arg c_0)}{m_k - m_0} \pmod{2\pi} \,\middle|\, m = 0, 1, \ldots, 2(m_k - m_0) - 1 \right\} \tag{19}$$

then we can express dsig as

$$\operatorname{dsig}(r) : \theta \in [0, 2\pi) \to \begin{cases} \frac{1}{\pi} sweep_r(\theta) - 1, & \theta \in T_r \setminus U_r \\ \left\lfloor \frac{1}{\pi} sweep_r(\theta) \right\rfloor, & \text{otherwise} \end{cases} \tag{20}$$

The next sub-section shows how straightforward analysis of a particular Descartes signature leads directly to the Roots of Unity. Sub-section 2.4 shows an analysis that is not nearly as successful at finding them.

2.3. **Example:** $1 - x^n$.

Define the polynomial p by

$$p(x) = 1 - x^n = 1 + e^{i\pi} x^n$$

Analysis in terms of explicit sweep formulas and so forth is unnecessarily complicated, so we will proceed using basic principles. We begin with the observation that, as p has only two terms, the exceptional component of dsig never applies, so that $\operatorname{dsig}(p)(\theta) = \lfloor \frac{1}{\pi} sweep_p(\theta) \rfloor$ for all θ.

Now, the auxiliary polynomial for p is

$$p_\theta(x) := 1 + e^{i(\pi + n\theta)} x^n$$

so that the sweep of p_θ is exactly the smaller angle in the complex plane determined by the positive x axis and the terminal ray in direction $\pi + n\theta$. That angle has measure less than π except when $n\theta$ is a multiple of 2π. As a result,

$$\operatorname{dsig}(p)(\theta) = \begin{cases} 1, & \theta = 2\pi k/n \text{ for } k = 0, 1, \ldots, n-1 \\ 0, & \text{otherwise} \end{cases}$$

The Descartes signature, then, distinguishes n equally-spaced angles (including 0) as possible arguments for roots of p, and the signature rules out the existence of any roots with multiplicity greater than 1. (See Figure 5.) Bolstered by the Fundamental Theorem of Algebra (p must have n roots) and the fact that each of the roots necessarily has modulus 1, the Descartes signature has exactly identified the n-th Roots of Unity.

Being derived solely from the arguments of the polynomial's coefficients, this Descartes signature is shared by every polynomial of the form $a - bx^n$, for positive a and b.

2.4. **Example:** $1 + x + x^2 + \cdots + x^n$.

Define the polynomial q by

$$q(x) = 1 + x + x^2 + \cdots + x^n$$

where $n > 0$. Note that $q(x) \cdot (1 - x) = 1 - x^{n+1}$; thus, q's roots should be all (and only) the $(n+1)$-th Roots of Unity except for 1. We shall see how well analysis of the Descartes signature does in getting us to those roots.

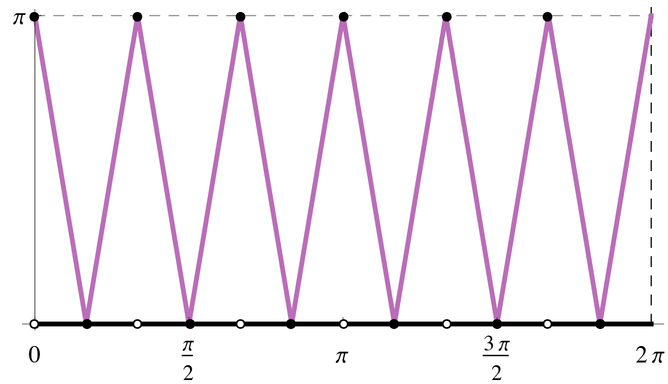


FIGURE 5. Graphs of $sweep_p$ (purple) and the (scaled-up) Descartes signature (black) for $p(x) := 1 - x^6$, showing the only non-zero Descartes bounds in the directions of the 6th Roots of Unity. (Signature values have been multiplied by π to better show the relationship with the sweep graph. The exceptional component of dsig is never used.) This signature is shared by all polynomials $a - bx^6$ for positive a and b.

With $\arg c_k = 0$ and $m_k = k$ for all k, we can click through various computations quickly:

$$sweep^+(q) = sweep^-(q) = 0$$

$$S_q = \{0, 0, \ldots, 0\} \quad (|S_q| = n) \qquad \sigma_p(0, \theta] = 0 \qquad \sigma_p[0, \theta) = n \;\; (\theta > 0)$$

$$sweep_q^+(\theta) = \theta n \qquad\qquad sweep_q(\theta) = \begin{cases} \theta n, & 0 \leq \theta < \pi \\ (2\pi - \theta)n, & \pi \leq \theta < 2\pi \end{cases}$$

$$T_q = \left\{\frac{\pi}{n} m\right\}_{m=0}^{2n-1} \qquad\qquad U_q = \bigcap_{k=1}^{n-1} \left\{\frac{\pi}{k} m\right\}_{m=0}^{2k-1} = \begin{cases} \varnothing, & n = 1 \\ \{0, \pi\}, & n > 1 \end{cases}$$

Now, shuffling the definitional components of $sweep_q$ with the definitional components of dsig yields

$$\text{(21)} \qquad \operatorname{dsig}(q)(\theta) = \begin{cases} \left\lfloor \frac{n}{\pi}\theta \right\rfloor & , \quad \theta = 0, \pi \\ \frac{n}{\pi}\theta - 1 & , \quad \theta = \frac{\pi}{n} m \;\middle|\; m = 1, 2, \ldots, n-1 \\ \left\lfloor \frac{n}{\pi}\theta \right\rfloor & , \quad \frac{\pi}{n}(m-1) < \theta < \frac{\pi}{n} m \;\middle|\; m = 1, 2, \ldots, n \\ \frac{n}{\pi}(2\pi - \theta) - 1 & , \quad \theta = \frac{\pi}{n}(2n - m) \;\middle|\; m = 1, 2, \ldots, n-1 \\ \left\lfloor \frac{n}{\pi}(2\pi - \theta) \right\rfloor & , \quad \frac{\pi}{n}(2n - m) < \theta < \frac{\pi}{n}(2n - m + 1) \;\middle|\; m = 1, 2, \ldots, n \end{cases}$$

(valid for all $n > 0$), which simplifies to

$$\text{(22)} \qquad \operatorname{dsig}(q)(\theta) = \begin{cases} 0 & , \quad \theta = 0 \\ \left\lceil \frac{n}{\pi}\theta \right\rceil - 1 & , \quad 0 < \theta < \pi \\ n & , \quad \theta = \pi \\ 2n - \left\lfloor \frac{n}{\pi}\theta \right\rfloor - 1 & , \quad \pi < \theta < 2\pi \end{cases}$$

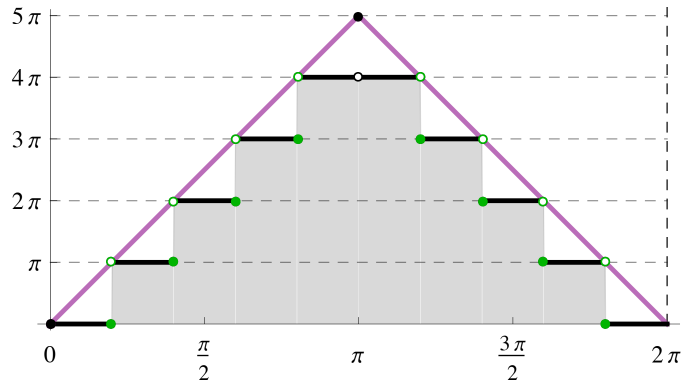


FIGURE 6. Graphs of $sweep_q$ (purple) and the (scaled-up) Descartes signature (black and green) for $q(x) := 1 + x + x^2 + \cdots + x^5$. (Signature values have been multiplied by π to better show the relationship with the sweep graph. Green dots indicate values at which the exceptional component of dsig is used.) This signature is shared by all 5th degree polynomials with a full set of five strictly positive coefficients.

This Descartes signature rules out roots in the direction $\theta = 0$ (corresponding to the obvious non-root 1), but does not conspicuously feature the remaining $(n + 1)$-th Roots of Unity that we know to be the roots of q. (Note that the signature does not —as it *should not*— rule out these other Roots of Unity, as their directions occur outside of the zero-height plateaux of the graph in Figure 6.)

Of course, such vaguaries are to be expected: the Descartes signature computed here is the signature, not only of our q, but of *every* n-th degree polynomial with a full set of strictly positive real coefficients. The signature must account for every possible arrangement of roots in such polynomials, not merely q's symmetrically-placed and multiplicity-1 Roots of Unity.

While not effective at narrowing the potential root pool to a finite collection, this signature provides a great deal of information about the polynomials it reflects: no positive real is a root (something we already knew), but in fact neither is any complex number with argument up to π/n; no number with argument up to $2\pi/n$ is a double-root, no number with argument up to $3\pi/n$ is a triple-root, and so forth; the only possibility for a root with multiplicity n is a negative real (something, again, we already knew).

We conclude by observing that this signature offers some potentially egregious overestimates on root counts, allowing for non-real roots with multiplicity up to $n - 1$. For $n > 2$, this is simply not possible within the given class of polynomials: the non-real roots of any polynomial with positive (or even merely *real*) coefficients must occur in conjugate pairs, so any non-real root can have multiplicity no greater than $n/2$.

3. Epilog

I have created an interactive Mathematica 6 notebook for investigating coefficient sweeps and the Descartes signature. (The figures in this paper come from that notebook.) I submitted the notebook to the Wolfram Demonstration Project[8] with the title "Descartes Signature Explorer" and will update this document with a proper citation when the WDP accepts it for publication. Interested parties can email me directly for the notebook file.

A note about computer-generated graphs of the Descartes signature: The condition distinguishing the components of the Descartes Signature formula (see Equations (17) and (20)) resists accurate plotting-by-random-sampling. The inherent imprecision of computing a number's argument, combined with the need to compare such arguments against the irrational π, provide an ever-present danger of erroneous evaluation of the condition, with a near-certainty that a handful of isolated "exceptional" values will not appear in the sampled set. With this in mind, it is generally preferable to pre-compute all points of discontinuity —via sets S_r, T_r, and U_r (see Equations (14), (18), and (19))— and to construct the piece-wise constant elements of the signature graph directly.

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[8] http://demonstrations.wolfram.com