THE LAWS OF COSINES FOR NON-EUCLIDEAN TETRAHEDRA

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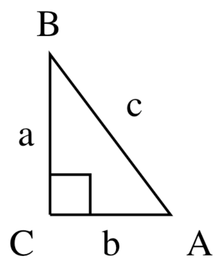
Darko Veljan's article "The 2500-Year-Old Pythagorean Theorem"[1] discusses the history and lore of "probably the only nontrivial theorem in mathematics that most people know by heart", depicted in Figure 1. The article motivated the author to put his own few-years-old results on public display.



$$c^2 = a^2 + b^2$$

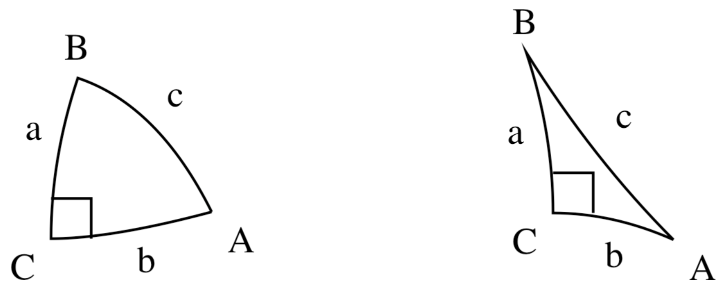
Figure 1. The Pythagorean Theorem for Euclidean Geometry

Veljan mentions a large number of relatives of "the Theorem" in various contexts. Of interest here are the direct analogues governing right triangles in spherical and hyperbolic geometry (Figure 2), the extensions that apply arbitrary triangles (Figure 3), and the generalization to Euclidean tetrahedra (Figure 4(a)). The "Law of Cosines" extensions of the last (Figure 4(b)), while perhaps not universally known, *are* known, are readily proven, and have straightforward analogues in all higher dimensions.

The purpose of this note is to formally present non-Euclidean counterparts of the two tetrahedral Laws of Cosines. The formulas appear below in Figure 5, with the rather elegant Pythagorean versions —for "right-cornered" tetrahedra (in which $m\angle BDC = m\angle CDA = m\angle ADB = \pi/2 = m\angle DA = m\angle DB = m\angle DC$)— in Figure 6. The author has posted these results a time or two to online discussion lists; the results seem to have been unknown to the mathematical community before then.

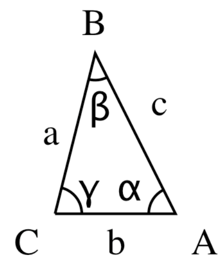
Date: 1 January, 2005; updated 30 January, 2006, to include the Second Law of Cosines.

[1]*Mathematics Magazine*, Vol. 73, No. 4. (October, 2000.)

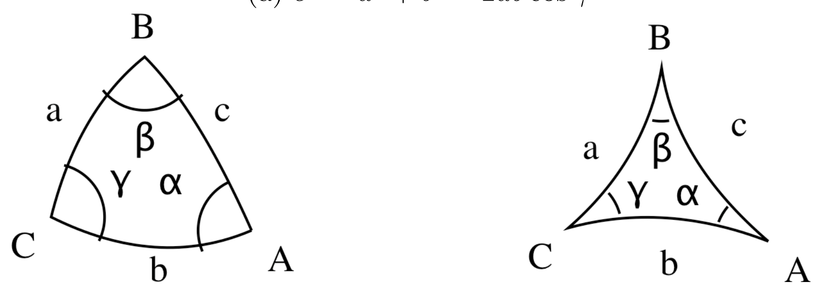


(a) $\cos c = \cos a \cos b$ (b) $\cosh c = \cosh a \cosh b$

FIGURE 2. The Pythagorean Theorems for (a) spherical and (b) hyperbolic geometry



(a) $c^2 = a^2 + b^2 - 2ab\cos\gamma$



(b)

$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$

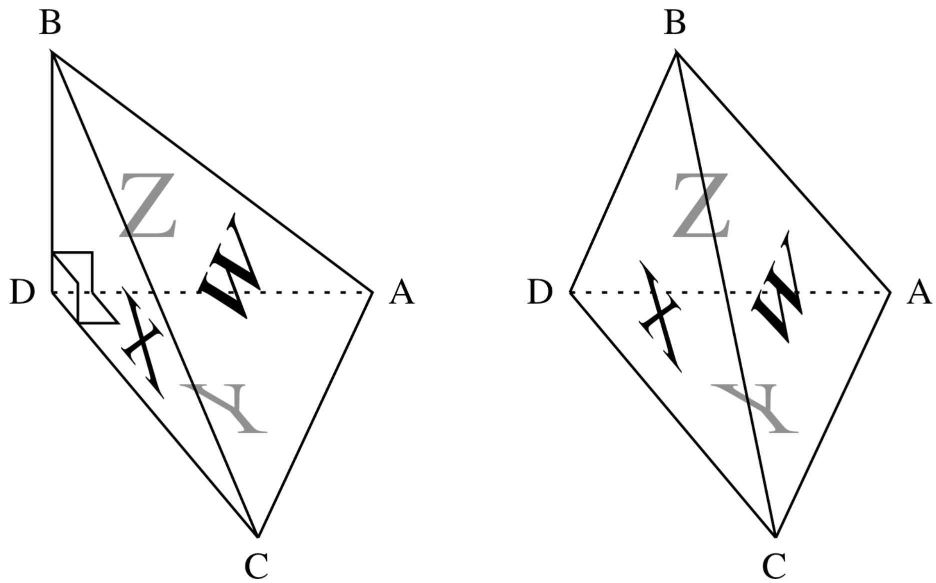
$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c$

(c)

$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$

$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cosh c$

FIGURE 3. Assorted Laws of Cosines



(a)

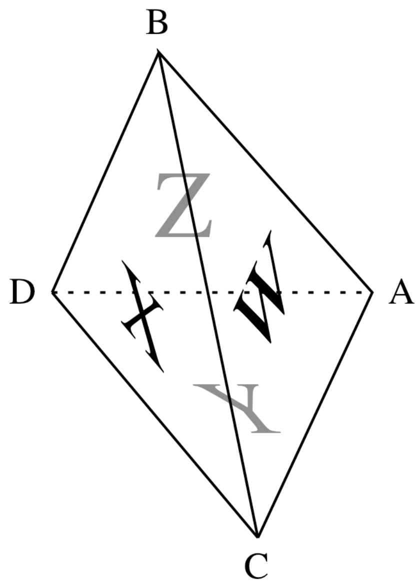
$$W^2 = X^2 + Y^2 + Z^2$$

(b)

$$\begin{aligned} W^2 &= X^2 + Y^2 + Z^2 - 2YZ\cos\angle DA \\ &\quad -2ZX\cos\angle DB - 2XY\cos\angle DC \end{aligned}$$

$$\begin{aligned} W^2 + X^2 &- 2WX\cos\angle BC \\ &= Y^2 + Z^2 - 2YZ\cos\angle AD \end{aligned}$$

FIGURE 4. The (a) Pythagorean Theorem and (b) Laws of Cosines for Euclidean Tetrahedra. (X, Y, Z, and W are face areas; "$\angle DA$", etc., are the dihedral angles between faces.)



(a) Spherical Space

$$
\begin{aligned}
\cos\tfrac{W}{2} &= \cos\tfrac{X}{2}\cos\tfrac{Y}{2}\cos\tfrac{Z}{2}\\
&+ \sin\tfrac{X}{2}\sin\tfrac{Y}{2}\sin\tfrac{Z}{2}S\\
&+ \cos\tfrac{X}{2}\sin\tfrac{Y}{2}\sin\tfrac{Z}{2}\cos\angle DA\\
&+ \sin\tfrac{X}{2}\cos\tfrac{Y}{2}\sin\tfrac{Z}{2}\cos\angle DB\\
&+ \sin\tfrac{X}{2}\sin\tfrac{Y}{2}\cos\tfrac{Z}{2}\cos\angle DC
\end{aligned}
$$

(b) Hyperbolic Space

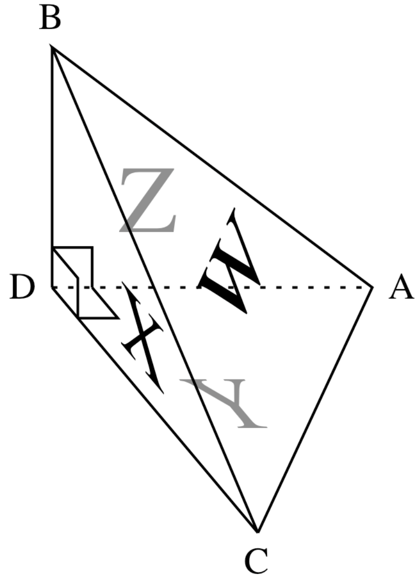
$$
\begin{aligned}
\cos\tfrac{W}{2} &= \cos\tfrac{X}{2}\cos\tfrac{Y}{2}\cos\tfrac{Z}{2}\\
&- \sin\tfrac{X}{2}\sin\tfrac{Y}{2}\sin\tfrac{Z}{2}S\\
&+ \cos\tfrac{X}{2}\sin\tfrac{Y}{2}\sin\tfrac{Z}{2}\cos\angle DA\\
&+ \sin\tfrac{X}{2}\cos\tfrac{Y}{2}\sin\tfrac{Z}{2}\cos\angle DB\\
&+ \sin\tfrac{X}{2}\sin\tfrac{Y}{2}\cos\tfrac{Z}{2}\cos\angle DC
\end{aligned}
$$

$$S := \sqrt{1 - 2\cos\angle DA\cos\angle DB\cos\angle DC - \cos^2\angle DA - \cos^2\angle DB - \cos^2\angle DC}$$

$$
\begin{aligned}
&\cos\tfrac{W}{2}\cos\tfrac{X}{2} - \sin\tfrac{W}{2}\sin\tfrac{X}{2}\cos\angle BC\\
=\ &\cos\tfrac{Y}{2}\cos\tfrac{Z}{2} - \sin\tfrac{Y}{2}\sin\tfrac{Z}{2}\cos\angle AD
\end{aligned}
$$

$$
\begin{aligned}
&\cos\tfrac{W}{2}\cos\tfrac{X}{2} + \sin\tfrac{W}{2}\sin\tfrac{X}{2}\cos\angle BC\\
=\ &\cos\tfrac{Y}{2}\cos\tfrac{Z}{2} + \sin\tfrac{Y}{2}\sin\tfrac{Z}{2}\cos\angle DA
\end{aligned}
$$

FIGURE 5. The Laws of Cosines for Non-Euclidean Tetrahedra



(a) Spherical Space

$$\cos\frac{W}{2} = \cos\frac{X}{2}\cos\frac{Y}{2}\cos\frac{Z}{2} + \sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}$$

(b) Hyperbolic Space

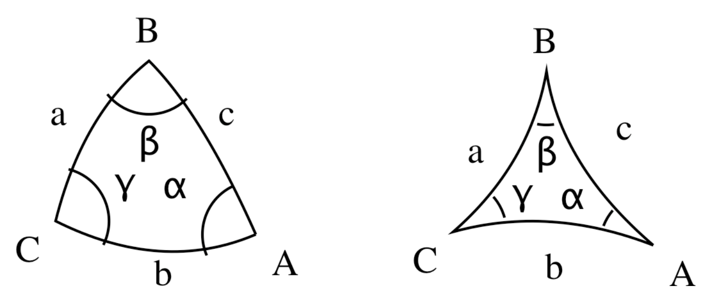
$$\cos\frac{W}{2} = \cos\frac{X}{2}\cos\frac{Y}{2}\cos\frac{Z}{2} - \sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}$$

FIGURE 6. The Pythagorean Theorems for Right-Cornered Non-Euclidean Tetrahedra

1. Preliminaries

Proof of these identities begins with the remarkably simple formulas of non-Euclidean area. See Figure 7.

- The area of a triangle in spherical geometry is equal to its "angular excess" (the amount by which sum is its radian angle measures exceeds π).
- The area of a triangle in hyperbolic geometry is equal to its "angular defect" (the amount by which the sum of its radian angle measures falls short of π).



Area $= (\alpha + \beta + \gamma) - \pi$ Area $= \pi - (\alpha + \beta + \gamma)$

Figure 7. Formulas for non-Euclidean Triangle Area

1.1. Notation.

To streamline the formulas, we'll employ a notational "Morse Code" —dots for cosines and dashes for sines— that merges the spherical and hyperbolic cases.

We will denote $\cos\theta$ —for angle θ— simply by $\ddot{\theta}$; complementarily, $\sin\theta$ appears as $\overline{\theta}$. For side x, we'll denote both $\cos x$ and $\cosh x$ by $\ddot{x}$, while both $\sin x$ and $\sinh x$ become $\overline{x}$. (There is no ambiguity here. The notation will indicate circular functions when representing the spherical case, and hyperbolic functions for the hyperbolic case.) To account for the occasional sign change, we agree that, in "$\pm$" and "$\mp$", the top sign applies to the spherical case, and the bottom one to the hyperbolic case. Then we have formulas such as these

$$\text{Law of Cosines for Sides} \quad \ddot{c} = \ddot{a}\ddot{b} \pm \overline{a}\overline{b}\ddot{\gamma}$$

$$\text{(solved for the angle} \quad \ddot{\gamma} = \pm\frac{\ddot{c} - \ddot{a}\ddot{b}}{\overline{a}\overline{b}} \quad)$$

$$\text{Area of Triangle} \quad \pm(\alpha + \beta + \gamma) \mp \pi$$

Also, we'll reference half-angles (and half-sides) often; we'll write x_2 for $\frac{x}{2}$ to save some typesetting space.

1.2. **A Heron-like Area Formula.** Our target formulas involve the cosine of the half-area of a triangle. This is easy to compute in terms of angles. For example,

$$\begin{aligned}
\cos\frac{W}{2} &= \cos\left(\frac{\left(\pm\left(\alpha+\beta+\gamma\right)\mp\pi\right)}{2}\right)\\
&= \cos\left(\pm\left(\alpha_2+\beta_2+\gamma_2\right)\mp\frac{\pi}{2}\right)\\
&= \sin\left(\alpha_2+\beta_2+\gamma_2\right)\\
&= \overline{\alpha_2}\ddot{\beta}_2\ddot{\gamma}_2+\ddot{\alpha}_2\overline{\beta_2}\ddot{\gamma}_2+\ddot{\alpha}_2\ddot{\beta}_2\overline{\gamma_2}-\overline{\alpha_2}\,\overline{\beta_2}\,\overline{\gamma_2}
\end{aligned}$$

The Law of Cosines for Sides allows us to convert this to a formula in terms of the triangle's (half-)sides, just as the Heron formula for Euclidean triangles computes area from sides. The reader can verify the following

$$\begin{aligned}
\ddot{\gamma}_2{}^2 &= \tfrac{1}{2}\left(1+\ddot{\gamma}\right)=\left(\overline{a_2+b_2+c_2}\right)\left(\overline{a_2+b_2-c_2}\right)/\overline{a}\overline{b} = \overline{s}\,\left(\overline{s-c}\right)/\overline{a}\overline{b}\\
\overline{\gamma_2}^2 &= \pm\tfrac{1}{2}\left(1-\ddot{\gamma}\right)=\left(\overline{-a_2+b_2+c_2}\right)\left(\overline{a_2-b_2+c_2}\right)/\overline{a}\overline{b}=\left(\overline{s-a}\right)\left(\overline{s-b}\right)/\overline{a}\overline{b}
\end{aligned}$$

with "semi-perimeter" $s := a_2+b_2+c_2 = \left(a+b+c\right)/2$.

The terms in the area formula then expand:

$$\begin{aligned}
\overline{\alpha_2}\ddot{\beta}_2\ddot{\gamma}_2 &= \sqrt{\tfrac{\left(\overline{s-b}\right)\left(\overline{s-c}\right)}{\overline{b}\overline{c}}\cdot\tfrac{\overline{s}\left(\overline{s-b}\right)}{\overline{c}\overline{a}}\cdot\tfrac{\overline{s}\left(\overline{s-c}\right)}{\overline{a}\overline{b}}} &&= \overline{s}\left(\overline{s-b}\right)\left(\overline{s-c}\right)/\overline{a}\overline{b}\overline{c}\\
\ddot{\alpha}_2\overline{\beta_2}\ddot{\gamma}_2 &= &&= \overline{s}\left(\overline{s-a}\right)\left(\overline{s-c}\right)/\overline{a}\overline{b}\overline{c}\\
\ddot{\alpha}_2\ddot{\beta}_2\overline{\gamma_2} &= &&= \overline{s}\left(\overline{s-a}\right)\left(\overline{s-b}\right)/\overline{a}\overline{b}\overline{c}\\
-\overline{\alpha_2}\,\overline{\beta_2}\,\overline{\gamma_2} &= &&= -\left(\overline{s-a}\right)\left(\overline{s-b}\right)\left(\overline{s-c}\right)/\overline{a}\overline{b}\overline{c}
\end{aligned}$$

Combining the terms, and canceling a denominator rewritten as $8\ddot{a}_2\overline{a_2}\ddot{b}_2\overline{b_2}\ddot{c}_2\overline{c_2}$, the sum simplifies to

$$\cos\frac{W}{2}=\frac{\ddot{a}_2{}^2+\ddot{b}_2{}^2+\ddot{c}_2{}^2-1}{2\ddot{a}_2\ddot{b}_2\ddot{c}_2}=\frac{1+\ddot{a}+\ddot{b}+\ddot{c}}{4\ddot{a}_2\ddot{b}_2\ddot{c}_2}$$

1.3. **The Heron-like Formula in Tetrahedral Context.** Taking our triangle, $\triangle ABC$, to be the face of a tetrahedron with fourth vertex D (see Figure 4), we can express the cosines of the triangle's half-edges (and ultimately its half-area) in terms of the lengths of half-edges and measures of angles of elements meeting at D:

$$\begin{array}{lll}
x := |DA| & y := |DB| & z := |DC|\\
\theta := m\angle BDC & \phi := m\angle CDA & \psi := m\angle ADB
\end{array}$$

The (triangular) Law of Cosines for Sides gives us

$$\begin{aligned}
\ddot{a} &= \ddot{y}\ddot{z}\pm\overline{yz}\ddot{\theta}\\
\ddot{b} &= \ddot{z}\ddot{x}\pm\overline{zx}\ddot{\phi}\\
\ddot{c} &= \ddot{x}\ddot{y}\pm\overline{xy}\ddot{\psi}
\end{aligned}$$

The Heron-like formula for W then becomes

$$\begin{aligned}\cos\frac{W}{2} &= \frac{1+\left(\ddot{y}\ddot{z}\pm\overline{yz}\ddot{\theta}\right)+\left(\ddot{z}\ddot{x}\pm\overline{zx}\ddot{\phi}\right)+\left(\ddot{x}\ddot{y}\pm\overline{xy}\ddot{\psi}\right)}{4\ddot{a}_2\ddot{b}_2\ddot{c}_2}\\ &= \frac{1+\ddot{y}\ddot{z}+\ddot{z}\ddot{x}+\ddot{x}\ddot{y}\pm\overline{yz}\ddot{\theta}\pm\overline{zx}\ddot{\phi}\pm\overline{xy}\ddot{\psi}}{4\ddot{a}_2\ddot{b}_2\ddot{c}_2}\end{aligned}$$

or in terms of half-sides

$$\cos\frac{W}{2} = \frac{\ddot{x}_2{}^2\ddot{y}_2{}^2\ddot{z}_2{}^2\pm\overline{x_2}^2\overline{y_2}^2\overline{z_2}^2\pm\overline{y_2}\ddot{y}_2\overline{z_2}\ddot{z}_2\ddot{\theta}\pm\overline{z_2}\ddot{z}_2\overline{x_2x_2}\ddot{\phi}\pm\overline{x_2}\ddot{x}_2\overline{y_2y_2}\ddot{\psi}}{\ddot{a}_2\ddot{b}_2\ddot{c}_2}$$

2. Verifying the Laws

2.1. **The First Law of Cosines.** Here, we will demonstrate that the right-hand side of our proposed First Law of Cosines for Non-Euclidean Tetrahedra

$$\begin{aligned}\cos\frac{W}{2} &= \cos\tfrac{X}{2}\cos\tfrac{Y}{2}\cos\tfrac{Z}{2}\pm\sin\tfrac{X}{2}\sin\tfrac{Y}{2}\sin\tfrac{Z}{2}S\\ &\quad+\cos\tfrac{X}{2}\sin\tfrac{Y}{2}\sin\tfrac{Z}{2}\cos\angle DA\\ &\quad+\sin\tfrac{X}{2}\cos\tfrac{Y}{2}\sin\tfrac{Z}{2}\cos\angle DB\\ &\quad+\sin\tfrac{X}{2}\sin\tfrac{Y}{2}\cos\tfrac{Z}{2}\cos\angle DC\end{aligned}$$

reduces to the formula at the end of the preceding subsection, thereby proving this Law.

Expanding the bits of the expression gives

$$\begin{aligned}\cos\frac{X}{2} &= \frac{1+\ddot{y}+\ddot{z}+\ddot{a}}{4\ddot{y}_2\ddot{z}_2\ddot{a}_2}=\frac{1+\ddot{y}+\ddot{z}+\ddot{y}\ddot{z}\pm\overline{yz}\ddot{\theta}}{4\ddot{y}_2\ddot{z}_2\ddot{a}_2}\\ &= \frac{(1+\ddot{y})(1+\ddot{z})\pm\overline{yz}\ddot{\theta}}{4\ddot{y}_2\ddot{z}_2\ddot{a}_2}=\frac{4\ddot{y}_2{}^2\ddot{z}_2{}^2\pm4\overline{y_2}\ddot{y}_2\overline{z_2}\ddot{z}_2\ddot{\theta}}{4\ddot{y}_2\ddot{z}_2\ddot{a}_2}\\ &= \frac{\ddot{y}_2\ddot{z}_2\pm\overline{y_2z_2}\ddot{\theta}}{\ddot{a}_2}\\ \cos\frac{Y}{2} &= \frac{\ddot{z}_2\ddot{x}_2\pm\overline{z_2x_2}\ddot{\phi}}{\ddot{b}_2}\\ \cos\frac{Z}{2} &= \frac{\ddot{x}_2\ddot{y}_2\pm\overline{x_2y_2}\ddot{\psi}}{\ddot{c}_2}\end{aligned}$$

and also

$$\sin\frac{X}{2}=\frac{\overline{y_2z_2}\overline{\theta}}{\ddot{a}_2}\qquad\sin\frac{Y}{2}=\frac{\overline{x_2z_2}\overline{\phi}}{\ddot{b}_2}\qquad\sin\frac{Z}{2}=\frac{\overline{y_2z_2}\overline{\psi}}{\ddot{c}_2}$$

Finally, the formulas for the dihedral angles at point D come from spherical trigonometry, with face-angles θ, ϕ, and ψ as the "sides", and dihedral angles $\angle DA$, $\angle DB$, and $\angle DC$ as the "angles". Thus,

$$\cos\angle DA = \frac{\ddot{\theta} - \ddot{\phi}\ddot{\psi}}{\overline{\phi\psi}} \qquad \cos\angle DB = \frac{\ddot{\phi} - \ddot{\theta}\ddot{\psi}}{\overline{\theta\psi}} \qquad \cos\angle DC = \frac{\ddot{\psi} - \ddot{\theta}\ddot{\phi}}{\overline{\theta\phi}}$$

and

$$\begin{aligned} S &= \sqrt{\begin{matrix} 1 - 2\cos\angle DA \cos\angle DB \cos\angle DC \\ -\cos^2\angle DA - \cos^2\angle DB - \cos^2\angle DC \end{matrix}} \\ &= \frac{1 + 2\ddot{\theta}\ddot{\phi}\ddot{\psi} - \ddot{\theta}^2 - \ddot{\phi}^2 - \ddot{\phi}^2}{\overline{\theta\phi\psi}} \end{aligned}$$

The terms of the proposed First Law of Cosines expansion are as follows:

$$\begin{aligned} \cos\frac{X}{2}\cos\frac{Y}{2}\cos\frac{Z}{2} &= \frac{\left(\ddot{y}_2\ddot{z}_2 \pm \overline{y_2 z_2}\ddot{\theta}\right)\left(\ddot{z}_2\ddot{x}_2 \pm \overline{z_2 x_2}\ddot{\phi}\right)\left(\ddot{x}_2\ddot{y}_2 \pm \overline{x_2 y_2}\ddot{\psi}\right)}{\ddot{a}_2\ddot{b}_2\ddot{c}_2} \\ &= \frac{\begin{pmatrix} \ddot{x}_2{}^2\ddot{y}_2{}^2\ddot{z}_2{}^2 \pm \overline{x_2}^2\overline{y_2}^2\overline{z_2}^2\ddot{\theta}\ddot{\phi}\ddot{\psi} \\ +\ddot{y}_2\overline{y_2}\ddot{z}_2\overline{z_2}\left(\pm\ddot{x}_2{}^2\ddot{\theta} + \overline{x_2}^2\ddot{\phi}\ddot{\psi}\right) \\ +\ddot{x}_2\overline{x_2}\ddot{z}_2\overline{z_2}\left(\pm\ddot{y}_2{}^2\ddot{\phi} + \overline{y_2}^2\ddot{\theta}\ddot{\psi}\right) \\ +\ddot{x}_2\overline{x_2 y_2}\ddot{y}_2\left(\pm\ddot{z}_2{}^2\ddot{\psi} + \overline{z_2}^2\ddot{\theta}\ddot{\phi}\right) \end{pmatrix}}{\ddot{a}_2\ddot{b}_2\ddot{c}_2} \end{aligned}$$

$$\begin{aligned} \pm\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}S &= \pm\frac{\overline{y_2 z_2}\,\overline{\theta}}{\ddot{a}_2}\cdot\frac{\overline{x_2 z_2}\,\overline{\phi}}{\ddot{b}_2}\cdot\frac{\overline{x_2 y_2}\,\overline{\psi}}{\ddot{c}_2}\cdot\frac{1 + 2\ddot{\theta}\ddot{\phi}\ddot{\psi} - \ddot{\theta}^2 - \ddot{\phi}^2 - \ddot{\psi}^2}{\overline{\theta\phi\psi}} \\ &= \frac{\pm\overline{x_2}^2\overline{y_2}^2\overline{z_2}^2\left(1 + 2\ddot{\theta}\ddot{\phi}\ddot{\psi} - \ddot{\theta}^2 - \ddot{\phi}^2 - \ddot{\psi}^2\right)}{\ddot{a}_2\ddot{b}_2\ddot{c}_2} \end{aligned}$$

$$\begin{aligned} \cos\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}\cos\angle DA &= \frac{\ddot{y}_2\ddot{z}_2 \pm \overline{y_2 z_2}\ddot{\theta}}{\ddot{a}_2}\cdot\frac{\overline{x_2 z_2}\,\overline{\phi}}{\ddot{b}_2}\cdot\frac{\overline{x_2 y_2}\,\overline{\psi}}{\ddot{c}_2}\cdot\frac{\ddot{\theta} - \ddot{\phi}\ddot{\psi}}{\overline{\phi\psi}} \\ &= \frac{\overline{x_2}^2\overline{y_2 z_2}\left(\ddot{y}_2\ddot{z}_2 \pm \overline{y_2 z_2}\ddot{\theta}\right)\left(\ddot{\theta} - \ddot{\phi}\ddot{\psi}\right)}{\ddot{a}_2\ddot{b}_2\ddot{c}_2} \\ &= \frac{\ddot{y}_2\overline{y_2}\ddot{z}_2\overline{z_2}\left(\overline{x_2}^2\ddot{\theta} - \overline{x_2}^2\ddot{\phi}\ddot{\psi}\right)}{\ddot{a}_2\ddot{b}_2\ddot{c}_2} \pm \frac{\overline{x_2}^2\overline{y_2}^2\overline{z_2}^2\left(\ddot{\theta}^2 - \ddot{\theta}\ddot{\phi}\ddot{\psi}\right)}{\ddot{a}_2\ddot{b}_2\ddot{c}_2} \\ \sin\frac{X}{2}\cos\frac{Y}{2}\sin\frac{Z}{2}\cos\angle DB &= \frac{\ddot{x}_2\overline{x_2}\ddot{z}_2\overline{z_2}\left(\overline{y_2}^2\ddot{\phi} - \overline{y_2}^2\ddot{\theta}\ddot{\psi}\right)}{\ddot{a}_2\ddot{b}_2\ddot{c}_2} \pm \frac{\overline{x_2}^2\overline{y_2}^2\overline{z_2}^2\left(\ddot{\phi}^2 - \ddot{\theta}\ddot{\phi}\ddot{\psi}\right)}{\ddot{a}_2\ddot{b}_2\ddot{c}_2} \\ \sin\frac{X}{2}\sin\frac{Y}{2}\cos\frac{Z}{2}\cos\angle DC &= \frac{\ddot{x}_2\overline{x_2}\ddot{y}_2\overline{y_2}\left(\overline{z_2}^2\ddot{\psi} - \overline{z_2}^2\ddot{\theta}\ddot{\phi}\right)}{\ddot{a}_2\ddot{b}_2\ddot{c}_2} \pm \frac{\overline{x_2}^2\overline{y_2}^2\overline{z_2}^2\left(\ddot{\psi}^2 - \ddot{\theta}\ddot{\phi}\ddot{\psi}\right)}{\ddot{a}_2\ddot{b}_2\ddot{c}_2} \end{aligned}$$

Therefore, combining the terms and invoking the trigonometric identity

$$\ddot{x} \pm \overline{x}^2 = \cos^2 x + \sin^2 x = \cosh^2 x - \sinh^2 x \equiv 1$$

gives

$$\begin{aligned} \text{Total} &= \left(\begin{array}{c} \ddot{x}_2{}^2 \ddot{y}_2{}^2 \ddot{z}_2{}^2 \pm \overline{x_2}^2 \overline{y_2}^2 \overline{z_2}^2 \\ \pm \ddot{y}_2 \overline{y_2} \ddot{z}_2 \overline{z_2} \ddot{\theta} \left(\ddot{x}_2{}^2 \pm \overline{x_2}^2 \right) \\ \pm \ddot{x}_2 \overline{x_2} \ddot{z}_2 \overline{z_2} \ddot{\phi} \left(\ddot{y}_2{}^2 \pm \overline{y_2}^2 \right) \\ \pm \ddot{x}_2 \overline{x_2} \ddot{y}_2 \overline{y_2} \ddot{\psi} \left(\ddot{z}_2{}^2 \pm \overline{z_2}^2 \right) \end{array} \right) / \ddot{a}_2 \ddot{b}_2 \ddot{c}_2 \\ &= \left(\begin{array}{c} \ddot{x}_2{}^2 \ddot{y}_2{}^2 \ddot{z}_2{}^2 \pm \overline{x_2}^2 \overline{y_2}^2 \overline{z_2}^2 \\ \pm \ddot{y}_2 \overline{y_2} \ddot{z}_2 \overline{z_2} \ddot{\theta} \pm \ddot{x}_2 \overline{x_2} \ddot{z}_2 \overline{z_2} \ddot{\phi} \pm \ddot{x}_2 \overline{x_2} \ddot{y}_2 \overline{y_2} \ddot{\psi} \end{array} \right) / \ddot{a}_2 \ddot{b}_2 \ddot{c}_2 \\ &= \cos \frac{W}{2} \end{aligned}$$

as desired.

2.2. **The Second Law of Cosines.**

Now we will verify the Second Law of Cosines for Non-Euclidean Tetrahedra

$$\cos \frac{W}{2} \cos \frac{X}{2} \pm \sin \frac{W}{2} \sin \frac{X}{2} \cos \angle BC \quad = \quad \cos \frac{Y}{2} \cos \frac{Z}{2} \pm \sin \frac{Y}{2} \sin \frac{Z}{2} \cos \angle DA$$

by exposing the symmetric nature of one side of the equation. Before doing so, we observe an immediate consequence of the relation.

Corollary 1. *In an ideal hyperbolic tetrahedron —that is, one with each of its vertices at infinity— dihedral angles along opposite angles are congruent.*

Proof. Simply note that each of the faces of the tetrahedron is an ideal triangle, with angles measuring 0 and hence areas W, X, Y, and Z measuring π. Each instance of the Second Law of Cosines then reduces to an equality between the cosines of opposing dihedral angles. □

Formulas from preceding sections give us the following

$$\begin{aligned} \cos \frac{Y}{2} \cos \frac{Z}{2} + \sin \frac{Y}{2} \cos \frac{Z}{2} \cos \angle DA &= \frac{1 + \ddot{x} + \ddot{z} + \ddot{b}}{4 \ddot{x}_2 \ddot{z}_2 \ddot{b}_2} \frac{1 + \ddot{x} + \ddot{y} + \ddot{c}}{4 \ddot{x}_2 \ddot{y}_2 \ddot{c}_2} + \frac{\overline{xz}\overline{\phi}}{4 \ddot{x}_2 \ddot{z}_2 \ddot{b}_2} \frac{\overline{xy}\overline{\psi}}{4 \ddot{x}_2 \ddot{y}_2 \ddot{c}_2} \frac{\overline{\theta} - \overline{\phi\psi}}{\overline{\phi\psi}} \\ &= \frac{(1 + \ddot{x} + \ddot{z} + \ddot{b})(1 + \ddot{x} + \ddot{y} + \ddot{c}) + \overline{x}^2 \overline{yz} (\ddot{\theta} - \ddot{\phi}\ddot{\psi})}{16 \ddot{x}_2{}^2 \ddot{y}_2 \ddot{z}_2 \ddot{b}_2 \ddot{c}_2} \end{aligned}$$

Now, using the (triangular) Law of Cosines in appropriate faces, we eliminate references to θ, ϕ, and ψ.

$$\begin{aligned} \overline{x}^2 \overline{yz} (\ddot{\theta} - \ddot{\phi}\ddot{\psi}) &= \overline{x}^2 (\overline{yz}\ddot{\theta}) - (\overline{xz}\ddot{\phi})(\overline{xy}\ddot{\psi}) = \pm \overline{x}^2 (\ddot{a} - \ddot{y}\ddot{z}) - (\ddot{b} - \ddot{x}\ddot{z})(\ddot{c} - \ddot{x}\ddot{y}) \\ &= (1 - \ddot{x}^2)(\ddot{a} - \ddot{y}\ddot{z}) - (\ddot{b} - \ddot{x}\ddot{z})(\ddot{c} - \ddot{x}\ddot{y}) \end{aligned}$$

so that

$$\begin{aligned}
&(1+\ddot{x}+\ddot{z}+\ddot{b})(1+\ddot{x}+\ddot{y}+\ddot{c})+\overline{x}^2\overline{yz}(\ddot{\theta}-\ddot{\phi}\ddot{\psi}) \\
&=(1+\ddot{x})^2+(1+\ddot{x})(\ddot{b}+\ddot{c}+\ddot{y}+\ddot{z})+\ddot{y}\ddot{z}+\ddot{c}\ddot{z}+\ddot{b}\ddot{y}+\ddot{b}\ddot{c} \\
&\quad +(1-\ddot{x}^2)(\ddot{a}-\ddot{y}\ddot{z})-\ddot{b}\ddot{c}+\ddot{b}\ddot{x}\ddot{y}+\ddot{c}\ddot{x}\ddot{z}-\ddot{x}^2\ddot{y}\ddot{z} \\
&=(1+\ddot{x})(1+\ddot{b}+\ddot{c}+\ddot{x}+\ddot{y}+\ddot{z}+\ddot{b}\ddot{y}+\ddot{c}\ddot{z}+(1-\ddot{x})(\ddot{a}-\ddot{y}\ddot{z}+\ddot{y}\ddot{z})) \\
&=2\ddot{x}_2{}^2(1+\ddot{a}+\ddot{b}+\ddot{c}+\ddot{x}+\ddot{y}+\ddot{z}-\ddot{a}\ddot{x}+\ddot{b}\ddot{y}+\ddot{c}\ddot{z})
\end{aligned}$$

Therefore,

$$\cos\frac{Y}{2}\cos\frac{Z}{2}+\sin\frac{Y}{2}\sin\frac{Z}{2}\cos\angle DA \quad = \quad \frac{1+\ddot{a}+\ddot{b}+\ddot{c}+\ddot{x}+\ddot{y}+\ddot{z}-\ddot{a}\ddot{x}+\ddot{b}\ddot{y}+\ddot{c}\ddot{z}}{8\ddot{y}_2\ddot{z}_2\ddot{b}_2\ddot{c}_2}$$

The right-hand side of this formula is symmetric with respect to switching pairs of edges

$$a \leftrightarrow x \qquad b \leftrightarrow y \qquad c \leftrightarrow c \qquad z \leftrightarrow z$$

which implies that the left-hand side must be symmetric with respect to switching its elements as well.

$$Y(=\triangle xbz) \leftrightarrow X(=\triangle ayz) \qquad W(=\triangle abc) \leftrightarrow Z(=\triangle xyc) \qquad \angle DA \leftrightarrow \angle BC$$

This proves the result.

2.3. **Pseudo-Faces.** As in the Euclidean case, the Second Law of Cosines invites the formal definition of "pseudo-faces" with areas H, J, and K satisfying the relations

$$\begin{aligned}
\cos\frac{W}{2}\cos\frac{X}{2}+\sin\frac{W}{2}\sin\frac{X}{2}\cos\angle BC &= \cos\frac{H}{2} = \cos\frac{Y}{2}\cos\frac{Z}{2}+\sin\frac{Y}{2}\sin\frac{Z}{2}\cos\angle DA \\
\cos\frac{W}{2}\cos\frac{Y}{2}+\sin\frac{W}{2}\sin\frac{Y}{2}\cos\angle CA &= \cos\frac{J}{2} = \cos\frac{Z}{2}\cos\frac{X}{2}+\sin\frac{Z}{2}\sin\frac{X}{2}\cos\angle DB \\
\cos\frac{W}{2}\cos\frac{Z}{2}+\sin\frac{W}{2}\sin\frac{Z}{2}\cos\angle AB &= \cos\frac{K}{2} = \cos\frac{X}{2}\cos\frac{Y}{2}+\sin\frac{X}{2}\sin\frac{Y}{2}\cos\angle DC
\end{aligned}$$

Unlike with the Euclidean case, I have not yet determined if the pseudo-faces have a geometric interpretation. (Whether they might provide any insights with respect to a Heron-like formula for volume is not at all clear.)

With the help of pseudo-faces, we can re-write First Law of Cosines in a symmetric, face-agnostic form that, as a bonus, avoids reference to the quantity S (or its counterparts in different orientations):

$$\begin{aligned}
0 \quad = \quad & 1-\ddot{W}_2{}^2-\ddot{X}_2{}^2-\ddot{Y}_2{}^2-\ddot{Z}_2{}^2-4\ddot{W}_2\ddot{X}_2\ddot{Y}_2\ddot{Z}_2 \\
& -\ddot{H}_2{}^2-\ddot{J}_2{}^2-\ddot{K}_2{}^2-2\ddot{H}_2\ddot{J}_2\ddot{K}_2 \\
& +2\ddot{H}_2(\ddot{W}_2\ddot{X}_2+\ddot{Y}_2\ddot{Z}_2)+2\ddot{J}_2(\ddot{W}_2\ddot{Y}_2+\ddot{Z}_2\ddot{X}_2)+2\ddot{K}_2(\ddot{W}_2\ddot{Z}_2+\ddot{X}_2\ddot{Y}_2)
\end{aligned}$$

We arrive at this formula by isolating S in the W-centric instance of the First Law of Cosines.

$$\overline{X_2Y_2Z_2}S=-\ddot{W}_2+\ddot{X}_2\ddot{Y}_2\ddot{Z}_2+\ddot{X}_2\overline{Y_2Z_2}\cos\angle DA+\overline{X_2}\ddot{Y}_2\overline{Z_2}\cos\angle DB+\overline{X_2Y_2}\ddot{Z}_2\cos\angle DC$$

After squaring both sides of the equation, we use the Second Law of Cosines to express $\cos\angle DA$, $\cos\angle DB$, and $\cos\angle DC$ in terms of the face and pseudo-face areas, and then simplify. (A computer algebra system comes in handy during this process.)

The instances of the First Law of Cosines can be recovered by making appropriate substitutions for H, J, and K, based on the Second Law of Cosines, and solving the resulting quadratic. Indeed, many additional formulas —for a total of 26, 244— arise from such substitutions.[2]

3. Remark

Infinitesimally, the Laws of Cosines for Non-Euclidean Tetrahedra become the Laws of Cosines for Euclidean Tetrahedra. This is demonstrated by expressing each half-area term with power series, and ignoring quantities with degree higher than two.

For the First Law, we have

$$\begin{aligned}
1-\frac{W^2}{8} \quad\sim\quad & \left(1-\frac{X^2}{8}\right)\left(1-\frac{Y^2}{8}\right)\left(1-\frac{Z^2}{8}\right) \pm \left(\frac{X}{2}\right)\left(\frac{Y}{2}\right)\left(\frac{Z}{2}\right) S \\
& + \left(1-\frac{X^2}{8}\right)\left(\frac{Y}{2}\right)\left(\frac{Z}{2}\right)\cos\angle DA \\
& + \left(\frac{X}{2}\right)\left(1-\frac{Y^2}{8}\right)\left(\frac{Z}{2}\right)\cos\angle DB \\
& + \left(\frac{X}{2}\right)\left(\frac{Y}{2}\right)\left(1-\frac{Z^2}{8}\right)\cos\angle DC \\
\sim\quad & \left(1-\frac{X^2}{8}-\frac{Y^2}{8}-\frac{Z^2}{8}\right) \pm (0)\, S \\
& + \frac{YZ}{4}\cos\angle DA + \frac{XZ}{4}\cos\angle DB + \frac{XY}{4}\cos\angle DC
\end{aligned}$$

whence

$$W^2 \quad\sim\quad X^2+Y^2+Z^2-2YZ\cos\angle DA-2XZ\cos\angle DB-2XY\cos\angle DC$$

Likewise, for the Second Law,

[2]Arithmetic check: There are 3 different ways to substitute into $\ddot{H}_2{}^2$ (using $\angle BC$ in both factors, using $\angle DA$ in both factors, or using each of these angles once); likewise, there are 3 ways each to substitute into $\ddot{J}_2{}^2$, $\ddot{K}_2{}^2$, $2\ddot{H}_2$, $2\ddot{J}_2$, $2\ddot{K}_2$. There are 36 ways to substitute into $2\ddot{H}_2\ddot{J}_2\ddot{K}_2$. Thus, the total number of formulas is $3^6 \cdot 36 = 26,244$.

$$\left(1-\frac{W^2}{8}\right)\left(1-\frac{X^2}{8}\right)+\frac{W}{2}\frac{X}{2}\cos\angle BC \quad\sim\quad \left(1-\frac{Y^2}{8}\right)\left(1-\frac{Z^2}{8}\right)+\frac{Y}{2}\frac{Z}{2}\cos\angle DA$$
$$\left(1-\frac{W^2}{8}-\frac{X^2}{8}\right)+\frac{WX}{4}\cos\angle BC \quad\sim\quad \left(1-\frac{Y^2}{8}-\frac{Z^2}{8}\right)+\frac{YZ}{4}\cos\angle DA$$
$$W^2+X^2-2WX\cos\angle BC \quad\sim\quad Y^2+Z^2-2YZ\cos\angle DA$$

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