WHAT IS THE PYTHAGOREAN THEOREM FOR RIGHT-CORNER SIMPLICES IN HYPERBOLIC 4-SPACE?

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In Euclidean any-dimensional space, our understanding of the Pythagorean Theorem for right-corner simplices is complete: always, *the content of the hypotenuse is equal to the sum of the squares of the contents of the legs.* In non-Euclidean space —in particular, hyperbolic space— our knowledge seems embarrassingly limited:

$$\begin{array}{lrcl}
\text{Dimension 2:} & \cosh \text{hyp} & = & \cosh \text{leg}_1 \cosh \text{leg}_2 \\[1ex]
\text{Dimension 3:} & \cos \dfrac{\text{hyp}}{2} & = & \cos \dfrac{\text{leg}_1}{2} \cos \dfrac{\text{leg}_2}{2} \cos \dfrac{\text{leg}_3}{2} - \sin \dfrac{\text{leg}_1}{2} \sin \dfrac{\text{leg}_2}{2} \sin \dfrac{\text{leg}_3}{2} \\[1ex]
\text{Dimension 4+:} & ??? & = & ???
\end{array}$$

The recently-revealed Derevnin-Mednykh formula for the volume of a hyperbolic tetrahedron [1] would seem to be key to conquering Dimension 4, yet I have had no success with it, even in the simplest of circumstances. Differential geometry isn't my field, and I'm at a loss as to how to proceed, but I thought I'd document what little progress I've made.

1. Isosceles 4-Simplices: How do they work?

Consider an *isosceles* right-corner 4-simplex: each of the four congruent leg-cells is itself an *isosceles* right-corner tetrahedron, with dihedral angles[1] $(A, B, C, D, E, F) = (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, D, D, D)$; the hypotenuse-cell is a *regular* tetrahedron, with congruent dihedral angles (G, G, G, G, G, G). One can verify that

$$1/3 \leq \quad \cos^2 D = x = \cos G \quad \leq 1/2$$

for x the parameter in integral formulas —derived from Derevnin-Mednykh— for the volumes, L and H, of the leg-cells and hypotenuse-cell:

$$\begin{aligned}
L &:= \frac{3}{2} \int_{1/3}^{x} \frac{1}{\sqrt{u(1-u)}} \operatorname{atanh}\sqrt{\frac{3u-1}{1-u}}\, \mathrm{d}u \\
H &:= 6 \int_{1/3}^{x} \frac{1}{\sqrt{1-u^2}} \operatorname{atanh}\sqrt{\frac{3u-1}{1-u}}\, \mathrm{d}u
\end{aligned}$$

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[1] Dihedral angles A, B, C are along edges that concur at a vertex; angles D, E, F are along edges opposite A, B, C, respectively.

The Pythagorean connection between these volumes is elusive. Indeed, the rather exotic values for $x = 1/2$ (i.e., a quadrupally-asymptotic simplex with triply-asymptotic legs)

$$L^\star := \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{\pi k}{2} = 0.45798\ldots \qquad H^\star := \sum_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{\pi k}{3} = 1.01494\ldots$$

suggest that the connection is quite non-trivial. Note that $L^\star$ is half of Catalan's constant, and $H^\star$ is known in mathematical literature (for instance, as the maximum of the Clausen function, Cl_2). And yet, Pythagoras beckons: surely, there *must* be an elegant connection.

Interestingly, H as a function of L looks very-nearly linear.[2] (See Figure 1.) Of course, given that the family of all isosceles simplices —from the infinitesimal to the asymptotic— have volumes bounded by just-over-1 and just-under-1/2, any variation is significant.

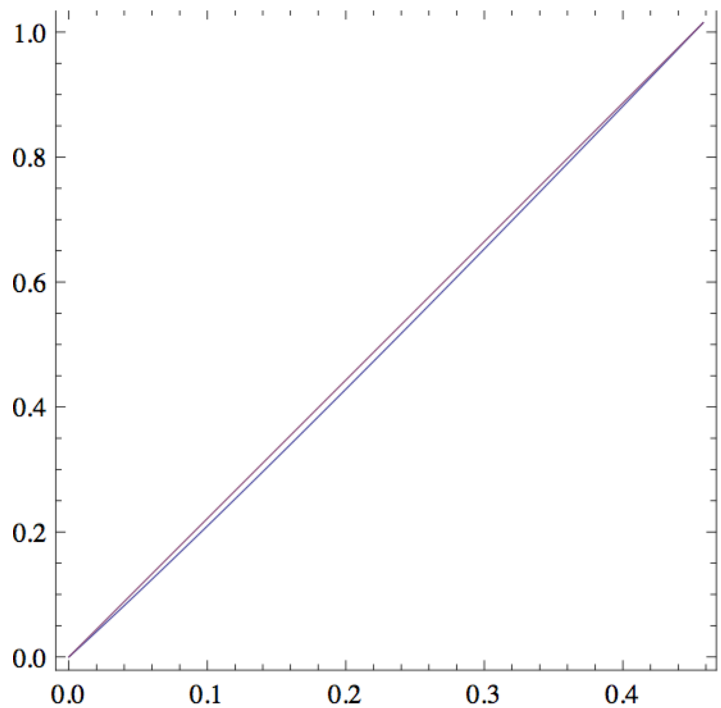

FIGURE 1. H as a function of L, and the line that approximates it. (I'm not sure which is which.)

References

[1] D. Derevnin and A. Mednykh, *A formula for the volume of a hyperbolic tetrahedron*, Communications of the Moscow Mathematical Society **60** No. 2 (2005), pg 346-348. `http://mathlab.snu.ac.kr/~top/articles/MednykhDerevninHyperbolicTetrahedron.pdf`

[2] In Euclidean space, the corresponding relation is *exactly* linear: $H^2 = L^2 + L^2 + L^2 + L^2 \implies H = 2L$.