ZIG-ZAG INVOLUTES, UP-DOWN PERMUTATIONS, AND SECANT AND TANGENT

B.D.S.MCCONNELL

In [2], Leo Gurin presents a delightful construction —attributed to Y.S.Chaikovsky— that exposes "the geometric meaning of every term in the [power] series" of the cosine and sine functions. This note adapts the method to demystify the corresponding series for the tangent and secant functions.

1. The Results

We summarize the Chaikovsky result as follows: Begin with an arc $I_1 := PP_1$ of length θ on a unit circle with center P_0. (We assume throughout that $0 < \theta < \pi/2$.) For each $i > 1$, let I_i be an involute of I_{i-1}, such that all curves I_i extend from a common endpoint P, as in Figure 1; for completeness, designate the radius PP_0 to be the "involute" I_0.

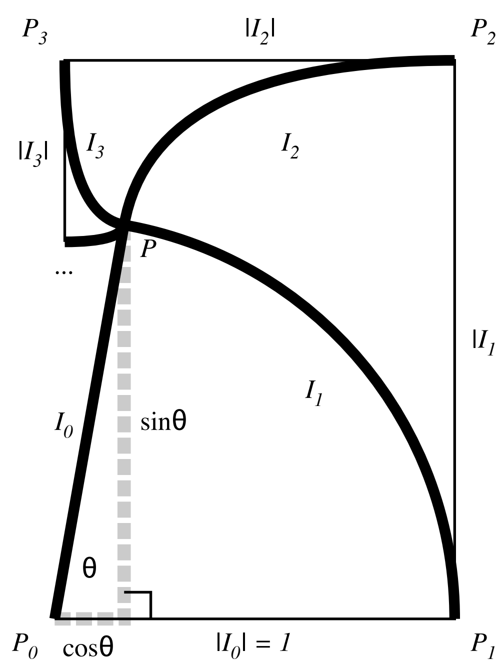

Figure 1. The Involute Pinwheel

As the polygonal spiral $P_0P_1P_2\ldots$ closes in on the point P, the terms of alternating series for $\cos\theta$ and $\sin\theta$ occur as lengths of the segments of that spiral (and, equivalently, lengths of the curves I_i):

$$\cos\theta = \sum_{i=0}^{\infty}(-1)^i \left|P_{2i}P_{2i+1}\right| = \sum_{i=0}^{\infty}(-1)^i \left|I_{2i}\right| \tag{1}$$

and

$$\sin\theta = \sum_{i=0}^{\infty}(-1)^i \left|P_{2i+1}P_{2i+2}\right| = \sum_{i=0}^{\infty}(-1)^i \left|I_{2i+1}\right| \tag{2}$$

Chaikovsky's arguments reveal that $|I_i| = \frac{1}{i!}\theta^i$, making (1) and (2) *power* series expansions.

With one simple conceptual change to Chaikovsky's approach —choosing involutes that form, not a pinwheel, but the zig-zag in Figure 2— we can devise comparable arguments to arrive at the power series for secant and tangent.

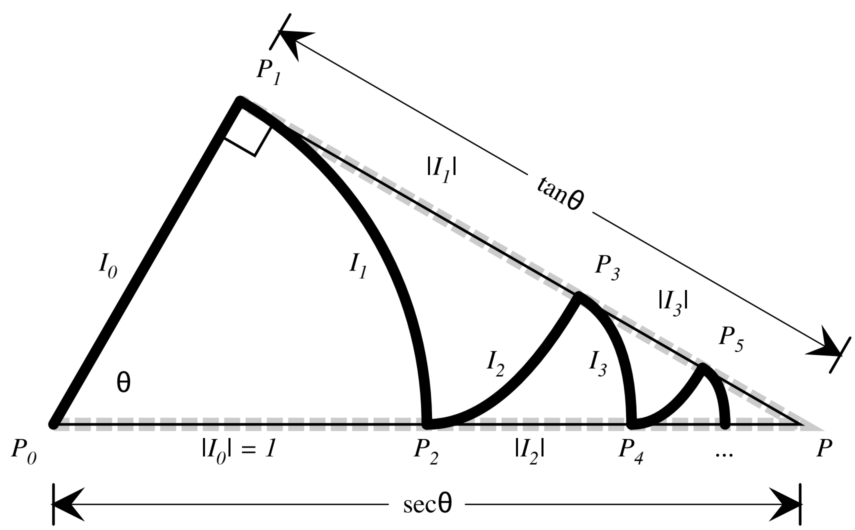

FIGURE 2. The Involute Zig-Zag.

Here, the points P_i converge on P, the point at which the extended radius P_0P_2 (containing all points P_{even}) meets the line (containing all points P_{odd}) tangent to the circle at P_1, and we have

$$\sec\theta = \sum_{i=0}^{\infty} \left|P_{2i}P_{2i+1}\right| = \sum_{i=0}^{\infty} \left|I_{2i}\right| \tag{3}$$

and

$$\tan\theta = \sum_{i=0}^{\infty} \left|P_{2i+1}P_{2i+2}\right| = \sum_{i=0}^{\infty} \left|I_{2i+1}\right| \tag{4}$$

Proof of the convergence is straightforward. We, therefore, focus on demonstrating that equations (5) and (6) are power series expansions with $|I_i| = \frac{1}{i!}z_i\theta^i$, where the z_i comprise a collection of known constants. Our arguments closely follow Chaikovsky's inspired lead, requiring nothing more than basic geometry and combinatorics, and an appeal to an elementary limit from calculus.

2. Polygonal Involutes

We will approach our result via polygonal approximations to our involutes. Driving our geometric construction will be combinatorial number sequences surprisingly appropriate to the zig-zag theme.

2.1. **Up-Down Permutations.** An *i-term up-down (or zig-zag) sequence of n elements* is a sequence $X = (x_1, x_2, \ldots, x_i)$ with $x_k \in \{1, 2, \ldots, n\}$ and $x_1 < x_2 > x_3 < \cdots x_i$. When $i = n$, we refer to X as an *i-term up-down permutation.* An up-down sequence $Xx := (x_1, x_2, \ldots, x_i, x)$ is called a *successor* of X.

By ordering the successors of X according to the difference between the final element of X and the appended element —that is, Xx comes before Xx' if and only if $|x - x_i| < |x' - x_i|$— we can deduce an important recursive formula for counting successors. First, partition the set $\{1, 2, \ldots, n\}$ into three disjoint sub-sets: the elements of X, the set $S := \{s|Xs \text{ is a successor of } X\}$, and a remainder set, R. The last element of X separates the elements of S from the elements of R on the number line. (See Figure 3.)

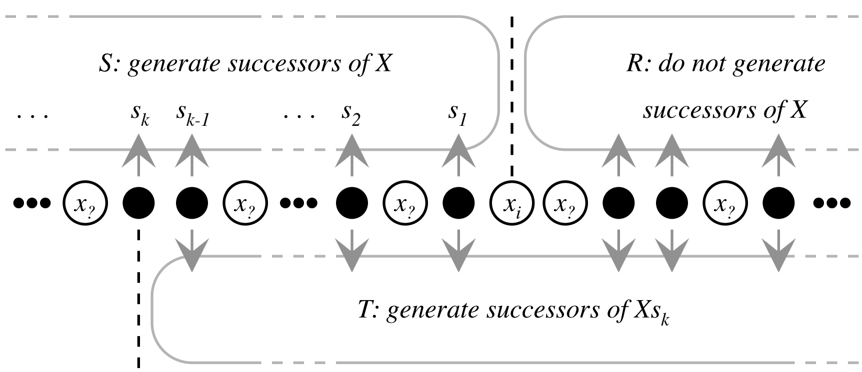

Figure 3. Recursive counting of successors: $|T| = n - |X| - |S| + (k - 1)$.

This, together with the ordering of S, implies that the sequence Xs_k has successors of the form Xs_kt, where $t \in T := R \cup \{s_1, s_2, \ldots, s_{k-1}\} = (\{1, 2, \ldots, n\} \backslash X \backslash S) \cup \{s_1, s_2, \ldots, s_{k-1}\}$. Therefore,

$$|\{\text{successors of the } k\text{-th successor of } X\}| = n - |X| - |\{\text{successors of } X\}| + (k - 1)$$

2.2. **Polygonal Involutes.** We start with a polygonal approximation of the arc I_1. For a given positive integer n, we divide the arc P_1P_2 into n congruent sub-arcs. Let I_1^n be the polygonal arc consisting of the chords subtended by those arcs; joining the endpoints of those chords to P_0, we have a collection of isosceles triangles with vertex angle θ/n. Define $u := 2\sin(\theta/2n)$ to be the length of each chord. Finally, proceeding from P_1 to P_2, label the segments of I_1^n with the numbers (more precisely, 1-term number sequences) from "(n)" down to "(1)" as in Figure 4.

Construction of polygonal approximations to the involutes proceeds as follows: Consider the j-th segment of I_i. For each sequence X on that segment, place the k-th successor of X (for all possible values of k) on a u^i-length extension of the $(j-k)$-th segment of I_i. (See Figures 4 and 5.) Having done this for all j, join the terminal endpoints of the accumulated extensions to create

I^n_{i+1}; transfer the sequences from the various extensions to congruent sub-segments (which, as we will show, happen to have length u^{i+1}) of the $n-i+1$ segments of I^n_{i+1}. (Again for completeness, we define I^n_0 to be the segment I_0.)

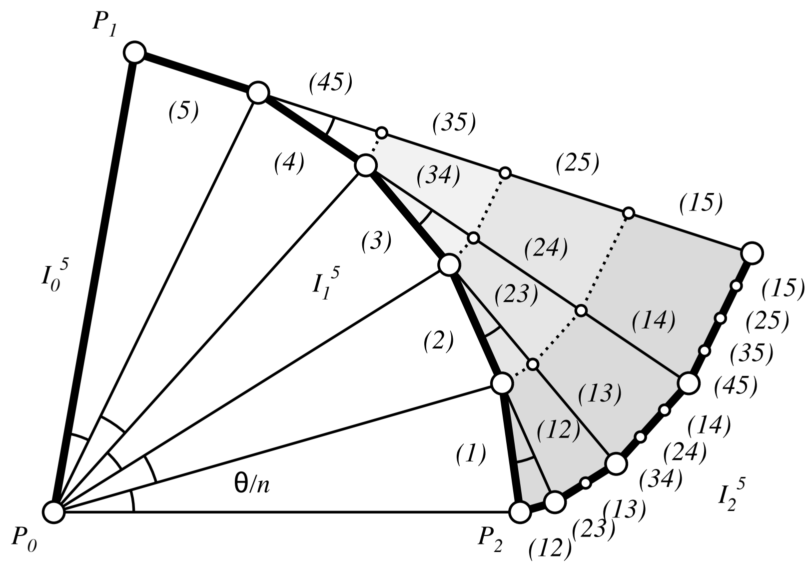

FIGURE 4. I^5_0, I^5_1, and I^5_2.

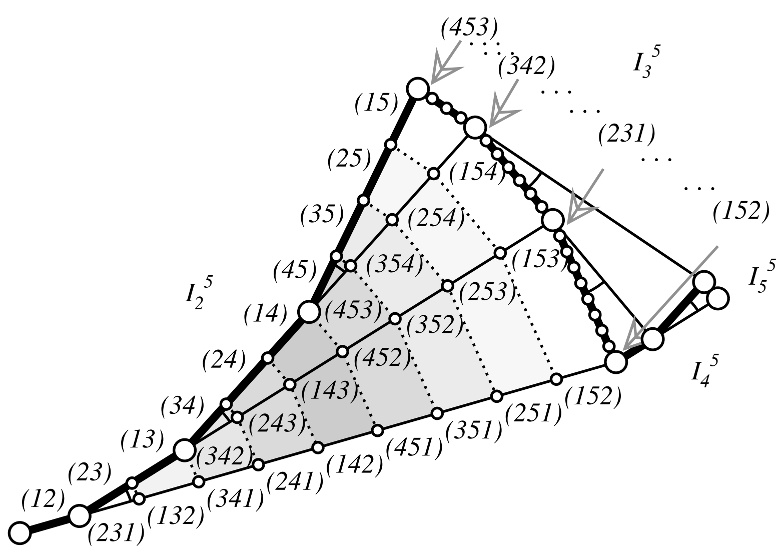

FIGURE 5. I^5_2, I^5_3, I^5_4, and I^5_5.

We observe that, for $i > 0$, I^n_{i+1} is the "polygonal involute of I^n_i" in the following sense: The extensions added to the j-th segment of I^n_i have (as we prove below) total length equal to the total length of "later" segments of I^n_i; thus, the corners of I^n_{i+1} track the path of an endpoint of I^n_i as I^n_i unfurls in discrete stages. Moreover, I^n_1 evidently approaches I_1 as n increases without bound; consequently, each I^n_i also approaches I_i.

We can prove the involute nature of I_i^n by showing that each sequence on the j-th segment of any I_i^n has exactly $j-1$ successors; this is certainly true for $i=1$. Assuming this property holds for a given i, consider sequences along extensions of the j-th segment: each extension sequence is the k-th successor of a sequence, X, on the $(j+k)$-th segment; each such X has $j+k-1$ successors, so its k-th successor, by the successor counting formula at the end of Section 2.1, has $n-(i)-(j+k-1)+(k-1) = n-i-j$ successors. Now, sequences along extensions of the j-th segment of I_i transfer to the j'-th segment of I_{i+1}^n, where $j+j' = n-i+1$. (Arithmetic check: extensions of the first segment —$j=1$— of I_i^n transfer to the last segment —$j' = n-(i+1)+1 = n-i$— of I_{i+1}^n, so that $j+j' = n-i+1$. As j increases, j' decreases, keeping a constant sum.) Thus, sequences along the j'-th segment of I_{i+1}^n have $n-i-j = n-i-(n-i+1-j') = j'-1$ successors, as required.

The involute nature of I_i^n implies that each segment of I_{i+1}^n is the base of an isosceles triangle with vertex angle θ/n and one leg composed of extensions to a particular segment of I_n^i. Since each extension has length u^i, we have that each sub-segment of a segment of I_{i+1}^n has length u^{i+1}.

3. Measuring the Polygonal Involutes, and Passing to the Limit

Our construction guarantees that, for all i, *the set of i-term up-down sequences of n elements is in one-to-one correspondence with the collection of u^i-length sub-segments along the polygonal curve I_i^n.* Thus,

$$|I_i^n| = u^i \cdot |\{i\text{-term up-down sequences of } n \text{ elements}\}|$$

Now, each i-term up-down sequence, X, of n elements can be associated with a unique i-term up-down *permutation* (of i elements): replace the smallest element in X with "1", replace the next-smallest element with "2", and so forth. Conversely, each i-term up-down permutation, Y, (of i elements) leads to $\binom{n}{i}$ i-term up-down sequences of n elements: for each choice of i elements from $\{1, 2, \ldots, n\}$, replace the "1" in Y with the smallest chosen element, replace "2" with the next-smallest element, and so forth. Therefore, we have

$$|I_i^n| = u^i \cdot \binom{n}{i} \cdot z_i, \qquad \text{where} \qquad z_i := |\{i\text{-term up-down permutations}\}|$$

The exact combinatorial nature of the numbers z_i —known as *(Euler) up-down, or zig-zag, numbers*— is irrelevant to us here. All that matters is that they are constants independent of n. This clears the way to take limits, exploiting this one elementary result:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

or, as we encounter it,

$$\lim_{n\to\infty} 2n \cdot \sin\left(\frac{\theta}{2n}\right) = \theta$$

Since each polygonal I_i^n tends to the corresponding curve I_i as n increases without bound, each polygonal length $|I_i^n|$ approaches the corresponding curve length $|I_i|$, and we compute as follows:

$$\begin{aligned} |I_i| &= \lim_{n\to\infty} |I_i^n| \\ &= \lim_{n\to\infty} u^i \cdot \binom{n}{i} \cdot z_i \\ &= \frac{1}{i!} \cdot z_i \cdot \prod_{j=0}^{i-1} \lim_{n\to\infty} \left(2n \sin\left(\frac{\theta}{2n}\right) \cdot \frac{n-i}{n} \right) \\ &= \frac{1}{i!} \cdot z_i \cdot \theta^i \end{aligned}$$

showing that our series (5) and (6) of $\sec\theta$ and $\tan\theta$ are in fact power series expansions.

4. Notes

- We have recovered the theorem of D. Andrë that $\sec\theta + \tan\theta$ is the exponential generating function for the numbers z_i. See [1] and [3].
- Chaikovsky's construction of polygonal approximations to the involute pinwheel lends itself to a reformulation: beginning with a polygonal arc with segments labelled "(1)" to "(n)", construction proceeds inductively by extending the segments to accommodate the "up-*up*" (that is, monotone increasing) successors of the sequences. (See Figures 6 and 7.)

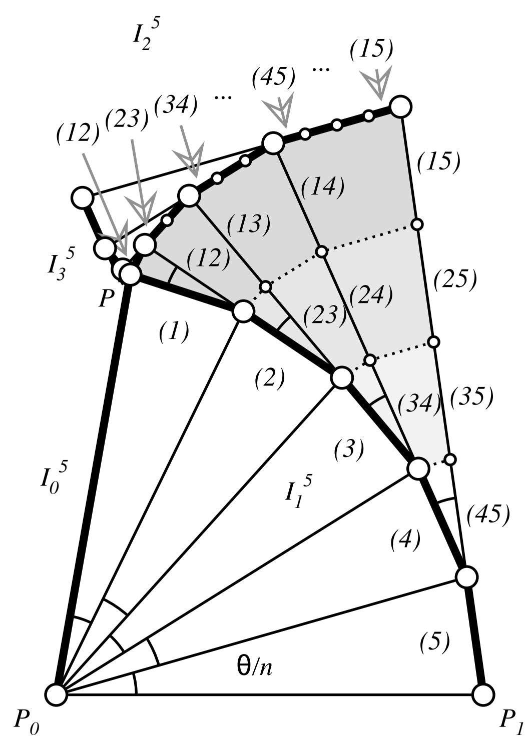

FIGURE 6. The Polygonal Involute Pinwheel, showing up-up sequences on I_1^5 and I_2^5.

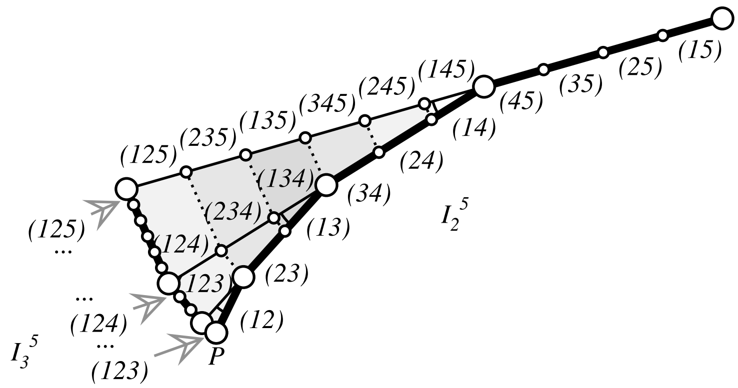

FIGURE 7. Up-up sequences on I_2^5 and I_3^5.

Just as in the zig-zag case (except with far less effort), one can show that all (and only) i-term up-up sequences of n elements occur on I_i^n. Just as in the zig-zag case, one can establish an $\binom{n}{i}$-to-1 correspondence between i-term up-up sequences of n elements and i-term up-up *permutations*. (This represents a minor —but satisfying— improvement over Chaikovsky's formulation: in the original approach, the binomial coefficient arises as the almost accidental result of an inductive computation; here, it arises in its most combinatorially meaningful capacity.) *Un*like in the zig-zag case, we have uniqueness: there is *exactly one* i-term up-up permutation, namely, "$(1, 2, \ldots, i)$"; there is no need, therefore, for an explicit "z_i"-like factor when counting the n-term sequences. Otherwise, all other computations of lengths of the (polygonal or true) involutes are completely analogous across both the pinwheel and zig-zag cases.

- The author gratefully acknowledges the assistance of Robin Chapman of the University of Exeter.
- Now, what about $\csc\theta$ and $\cot\theta$?

REFERENCES

[1] H. Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, 1965.
[2] Leo S. Gurin, A problem, *American Mathematical Monthly*, 103, 1996, 683-686.
[3] Ross Honsberger, *Mathematical Gems III*, MAA, 1985.

B.D.S.McConnell
math@daylateanddollarshort.com
3210 Elm Drive
Temple, TX 76502
USA