ZIG-ZAG INVOLUTES, UP-DOWN PERMUTATIONS, AND SECANT AND TANGENT

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In [3], Leo Gurin presents a delightful construction —attributed to Y. S. Chaikovsky— that reveals "the geometric meaning of every term in the [power] series" of the cosine and sine functions. This note adapts the method to demystify the corresponding series for the secant and tangent functions.

1. The Pinwheel and the Zig-Zag

Chaikovsky's construction begins with a circular arc, $I_1 := \widehat{PP_1}$, having center P_0, radius 1, and length θ. (We assume throughout that $0 < \theta < \pi/2$.) For each $i > 1$, let I_i be the involute of I_{i-1} that emerges from endpoint P, creating the pinwheel arrangement in Figure 1. For completeness, we designate the radius $\overline{PP_0}$ to be I_0.

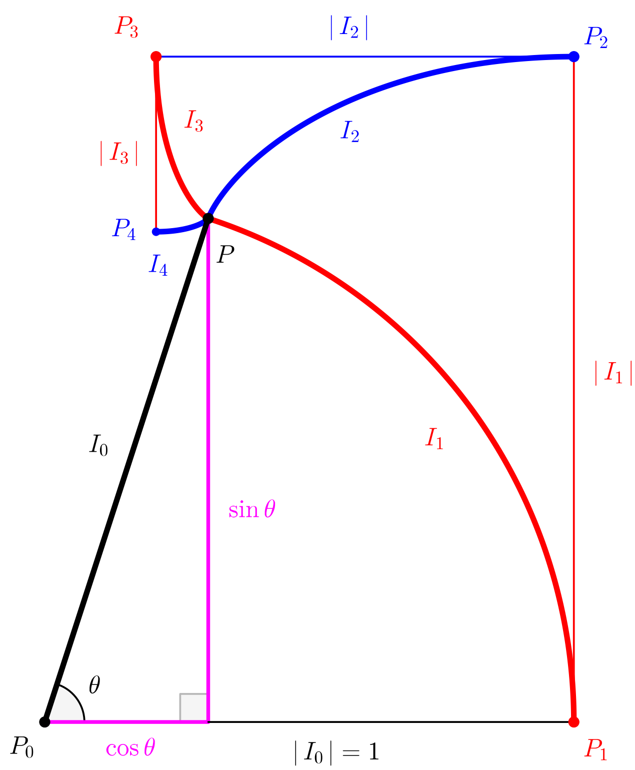


Figure 1. Chaikovsky's Involute Pinwheel

Date: 27 November, 2005. Revised 20 October, 2019.

Elementary trigonometry tells us that $\cos\theta$ and $\sin\theta$ are the lengths of, respectively, the horizontal and vertical legs of the right triangle with hypotenuse $\overline{PP_0}$. We observe that, as the polygonal spiral $P_0P_1P_2\ldots$ closes-in on the point P, the terms of alternating series for those values appear as the lengths of the horizontal and vertical segments of that spiral (equivalently, as lengths of the curves I_i):

$$\cos\theta = \sum_{j=0}^{\infty}(-1)^j\,|P_{2j}P_{2j+1}| = \sum_{j=0}^{\infty}(-1)^j\,|I_{2j}| = \sum_{i\,\text{even}} \pm|I_i| \tag{1}$$

$$\sin\theta = \sum_{j=0}^{\infty}(-1)^j\,|P_{2j+1}P_{2j+2}| = \sum_{j=0}^{\infty}(-1)^j\,|I_{2j+1}| = \sum_{i\,\text{odd}} \pm|I_i| \tag{2}$$

Chaikovsky uses clever combinatorial arguments show that $|I_i| = \frac{1}{i!}\theta^i$, making (1) and (2) the *power* series expansions.

With one simple conceptual change to Chaikovsky's approach —trading involutes that emerge from a common starting point to ones that emerge from successive terminal points— the pinwheel becomes the zig-zag in Figure 2.

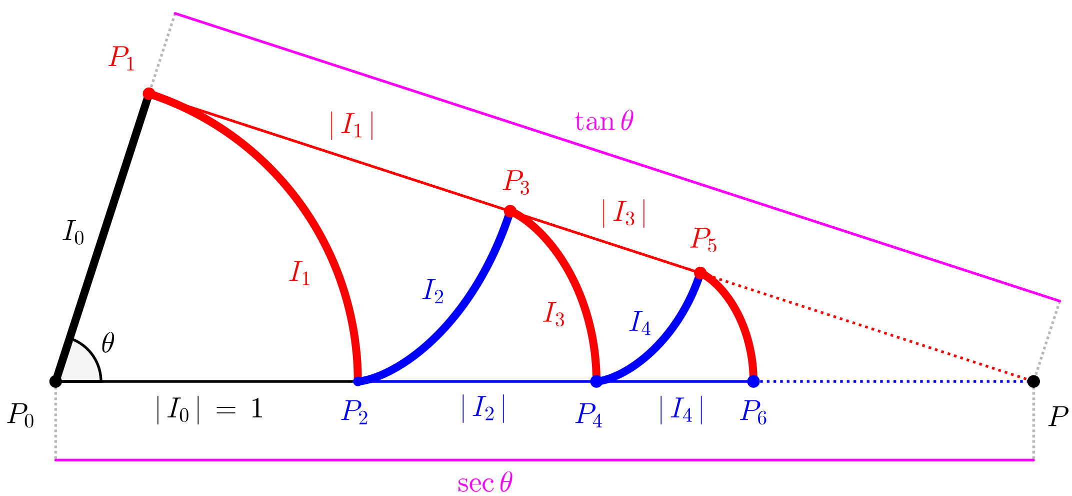


FIGURE 2. The Involute Zig-Zag

Here, I_1 is the circular arc $\overset{\frown}{P_1P_2}$ with center P_0, radius 1, and length θ; and I_{i+1} is the involute of I_i with endpoints P_i and P_{i+1}. (We designate radius $\overline{P_0P_1}$ to be I_0.) Points P_i converge on P, the point at which the extended radius $\overline{P_0P_2}$ (containing all points P_{even}) meets the circular arc's tangent line at P_1 (containing all points P_{odd}), and we have

$$\sec\theta := \left|\overline{P_0P}\right| = \sum_{j=0}^{\infty}|P_{2j}P_{2j+1}| = \sum_{j=0}^{\infty}|I_{2j}| = \sum_{i\,\text{even}} |I_i| \tag{3}$$

$$\tan\theta := \left|\overline{P_1P}\right| = \sum_{j=0}^{\infty}|P_{2j+1}P_{2j+2}| = \sum_{j=0}^{\infty}|I_{2j+1}| = \sum_{i\,\text{odd}} |I_i| \tag{4}$$

Proof of the convergence is straightforward and left to the reader. We focus on demonstrating that (3) and (4) are power series expansions, with $|I_i| = \frac{1}{i!} z_i \theta^i$, where the z_i comprise a collection of known constants. Our arguments closely follow Chaikovsky's inspired lead, requiring nothing more than the same basic geometry and slightly-more-elaborate combinatorics, and an appeal to an elementary limit from calculus.

2. Polygonal Involutes

We approach our result via increasingly-accurate polygonal approximations to our zig-zag's constituent involutes, extending the notion of such an approximation of a circular arc.

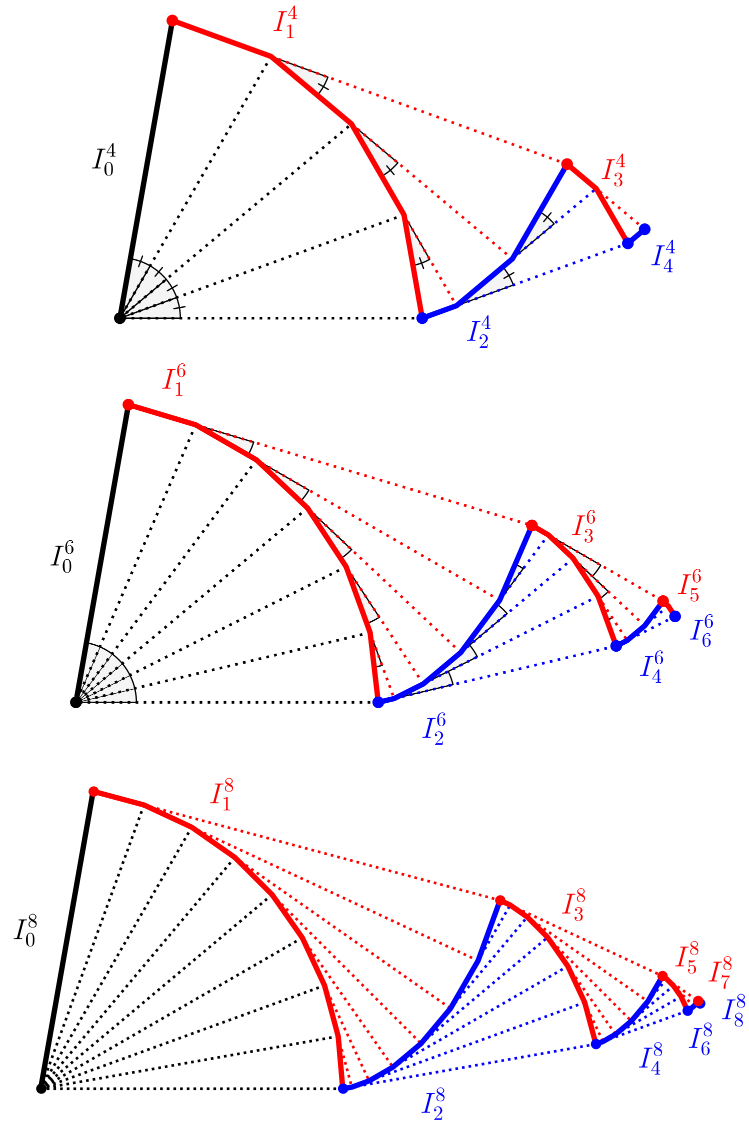


Figure 3. Polygonal zig-zag involutes I_i^4, I_i^6, and I_i^8.

Specifically, for integer $n > 1$, define I_0^n as the unit radius segment I_0 itself, and define I_1^n as the approximation of circular arc I_1 by n congruent chords, which we'll call *segments* of I_1^n; each segment is the base of an isosceles triangle with unit-length legs and vertex angle θ/n, so its length is $\lambda := 2\sin(\theta/2n)$. Now, iterate: for every $i < n$, extend each segment (except the last) of I_i^n by the total length of the *later* segments in I_i^n, as indicated in Figure 3. The extensions yield $n - i + 1$ isosceles triangles with vertex angle θ/n, and the bases of these triangles comprise the *segments* of I_{i+1}^n; note that each segment's length must be an integer multiple of λ^i.

The construction results in a total of $n + 1$ polygonal paths. Path I_{i+1}^n (for $i > 1$) is the "polygonal involute" of I_i^n in the sense that the endpoints of the segments of I_{i+1}^n mark the progress of I_i^n as it straightens in discrete stages. The distance from the starting point of I_i^n to the endpoint of I_{i+1}^n is the length of the fully-straightened path; that is, $|I_i^n|$.

As n increases without bound, this polygonal zig-zag acquires evermore zigs and zags, the vertex angles of the various isosceles triangles converge to 0, and the discrete involute-like behavior approaches continuous involute behavior, ensuring that each individual I_i^n converges to its counterpart I_i, and thus also that each $|I_i^n|$ converges to $|I_i|$.

2.1. **Measuring Polygonal Involutes, and Passing to the Limit.** As noted, the length of each segment of I_i^n —and thus of each complete I_i^n— is some integer multiple of λ^i. But *which* integers?

	$n = 4$		$n = 6$		$n = 8$	
i	seg lengths*	total	seg lengths*	total	segment lengths*	total
0	1	$1 = 1\binom{4}{0}$	1	$1 = 1\binom{6}{0}$	1	$1 = 1\binom{8}{0}$
1	$1, 1, 1, 1$	$4 = 1\binom{4}{1}$	$1, 1, 1, 1, 1, 1$	$6 = 1\binom{6}{1}$	$1, 1, 1, 1, 1, 1, 1, 1$	$8 = 1\binom{8}{1}$
2	$1, 2, 3$	$6 = 1\binom{4}{2}$	$1, 2, 3, 4, 5$	$15 = 1\binom{6}{2}$	$1, 2, 3, 4, 5, 6, 7$	$28 = 1\binom{8}{2}$
3	$3, 5$	$8 = 2\binom{4}{3}$	$5, 9, 12, 14$	$40 = 2\binom{6}{3}$	$7, 13, 18, 22, 25, 27$	$112 = 2\binom{8}{3}$
4	5	$5 = 5\binom{4}{4}$	$14, 26, 35$	$75 = 5\binom{6}{4}$	$27, 52, 74, 92, 105$	$350 = 5\binom{8}{4}$
5	—	—	$35, 61$	$96 = 16\binom{6}{5}$	$105, 197, 271, 323$	$896 = 16\binom{8}{5}$
6	—	—	61	$61 = 61\binom{6}{6}$	$323, 594, 791$	$1708 = 61\binom{8}{6}$
7	—	—	—	—	$791, 1385$	$2176 = 272\binom{8}{7}$
8	—	—	—	—	1385	$1385 = 1385\binom{8}{8}$

*when multiplied by λ^i, where $\lambda := 2\sin(\theta/2n)$

TABLE 1. Lengths of segments in the polygonal zig-zag involutes I_i^n

Row $i = 1$ of Table 1, being populated with 1s, records that the segments of I_1^n have equal lengths. Thereafter, the elements of row i are the first $n - i + 1$ partial sums of the *reversed-order* row $i - 1$. The sequences may appear somewhat haphazard, but their *totals* exhibit a pattern: each is the product of a naturally-corresponding binomial coefficient and, perhaps-surprisingly, a term of a *common* sequence of integers, $(z_i) := 1, 1, 1, 2, 5, 16, 61, \ldots$. We show in Section 3 that these z_i truly do what they seem to do (and, in particular, that they are independent of n); for now, we *assume* this fact, which allows us to write:

$$|I_i^n| = z_i \lambda^i \binom{n}{i} \tag{5}$$

At this point, we make our sole appeal to calculus, invoking a fundamental result:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \text{or, as we require it:} \qquad \lim_{n\to\infty}\lambda n=\lim_{n\to\infty}2n\cdot\sin\frac{\theta}{2n}=\theta \tag{6}$$

This catalyzes a straightforward calculation:

$$\begin{aligned} |I_i| := \lim_{n\to\infty}|I_i^n| &= \lim_{n\to\infty} z_i\lambda^i\binom{n}{i} \\ &= \frac{1}{i!}\cdot z_i\cdot\lim_{n\to\infty}(\lambda n)^i\cdot\lim_{n\to\infty}\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdot\dots\cdot\frac{n-i+1}{n} \\ &= \frac{1}{i!}\cdot z_i\cdot\theta^i \end{aligned} \tag{7}$$

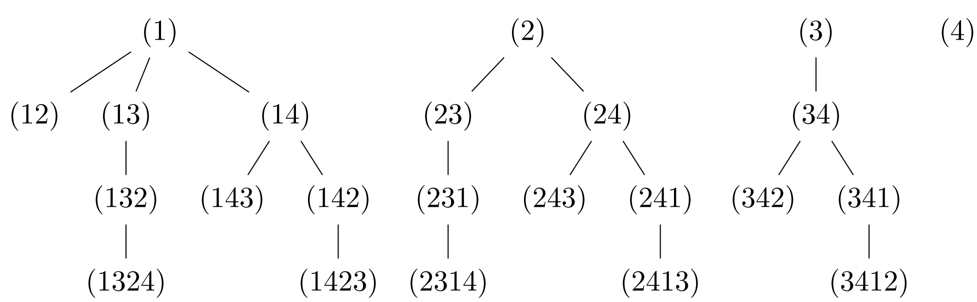
Just like that, we have demonstrated that the series (3) and (4) for $\sec\theta$ and $\tan\theta$ are, in fact, *power* series expansions, as claimed. □

We turn to investigating the constants z_i, which arise from yet-another zig-zaggy context.

3. Up-Down Sequences and Permutations

Definition. *An i-term* up-down sequence of n elements[1] *is a sequence $X=(x_1,x_2,\dots,x_i)$ with $x_k\in[n]:=\{1,2,\dots,n\}$ and $x_1<x_2>x_3<\cdots\lessgtr x_i$. When $i=n$, the sequence is known as an* up-down permutation *of $[n]$. The up-down sequence $Xs:=(x_1,x_2,\dots,x_i,s)$ is a* successor *of X, with s called a* successory *to X; successors are ordered so that, for $k\geq 1$, the k-th* successor *of X ends in the successory k-th closest to x_i (i.e., the k-th* successory*).*

For example, these are the up-down sequences of [4], each linked to its successor(s):



Observe that the ordering of successories allows us state that, if s_k is the k-th successory of X, then the successories of Xs_k are all of the *non*-successories of X, along with the *first* $k-1$ successories of X. (See Figure 4.)

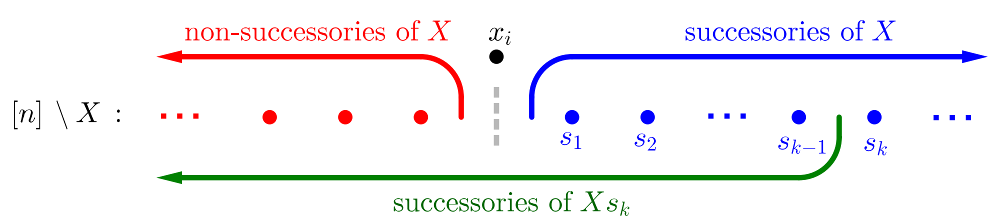


Figure 4. The successories of the k-th successor of X.

[1] Beware: In the literature, the similar term *zig-zag* sequence sometimes means *up-down or down-up*.

This gives a successor-counting recursion we'll find helpful:

$$
\begin{aligned}
(8)\qquad |\,\{\text{successors of } k\text{-th successor of } X\}\,| &= |\,\{\text{non-successories of } X\}\,| + (k-1) \\
&= |\,[n] \setminus X \setminus \{\text{successories of } X\}\,| + (k-1) \\
&= n - |X| - |\,\{\text{successors of } X\}\,| + (k-1)
\end{aligned}
$$

For another key counting result, consider: each i-term up-down sequence of $[n]$ corresponds to a unique i-term up-down *permutation* (of $[i]$): simply replace the k-th smallest element of the sequence with "k" for each k. Conversely, each i-term up-down permutation (of $[i]$) corresponds to $\binom{n}{i}$ distinct i-term up-down sequences of $[n]$: for each choice of i elements from $[n]$, replace "k" in the permutation with the k-th smallest chosen element. Therefore, we can write

$$
(9)\qquad |\{i\text{-term up-down sequences of } [n]\}| \;=\; \binom{n}{i}\, |\{i\text{-term up-down permutations}\}|
$$

Désiré André [1] first studied up-down permutations in 1879, finding that the number of i-term permutations, for the first few i, are:[2] (1,) 1, 1, 2, 5, 16, 61, 272, 1385, ... The reader may recognize these values as the z_i we encountered when measuring our polygonal involutes in Section 2.1. Indeed, the appearance of the binomial coefficient factor, both then and now, invites reinterpreting (9) to assert

$$
(10)\qquad |I_i^n| \;=\; \lambda^i\; |\{i\text{-term up-down sequences of } [n]\}|
$$

If we can verify this relation, then we will have shown that André's sequence actually matches our z_i. This, in turn, will close the gap in the argument of Section 2.1.

3.1. **Up-Downs on the Zig-Zag.** The clearest way to demonstrate (10) is to incorporate up-down sequences into the construction of our zig-zag involutes. We start with the "empty" sequence on I_0^n, and then each 1-term sequence (in "decreasing order" when traversing from P_1 to P_2) on each λ-length segment of I_1^n. (See Figure 5.)

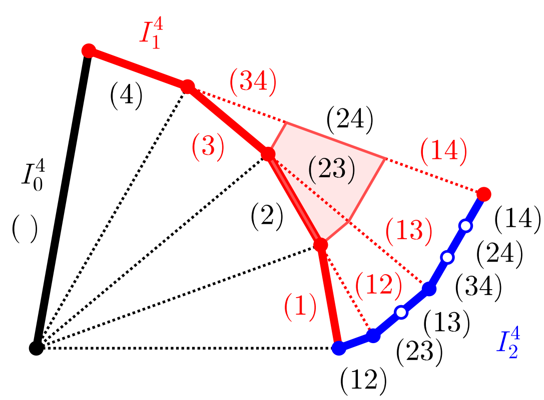


FIGURE 5. Adorning I_0^4, I_1^4, and I_2^4 with up-down sequences of [4].

[2] With the inclusion of the leading 1 for $i = 0$ —and representing the "empty" permutation— this is entry A000111 in the *On-line Encyclopedia of Integer Sequences.* `https://oeis.org/A000111`

Then, for each segment of I_1^n, we place its sequence's k-th successor on a λ-length extension of the k-th prior segment. Since each sequence has as many successors as its segment has priors, the extensions accumulate into the legs of the isosceles triangles from our previous construction strategy; this allows us to *transfer* the complete set of 2-term up-down sequences of $[n]$ from the legs of those triangles to the λ^2-length *sub*-segments of the corresponding bases, which comprise the segments of I_2^n.

Now, we iterate, placing the k-th successor of each i-term sequences from (a sub-segment of) each segment of I_i^n onto a λ^i-length extension of the k-th prior segment. Provided that the extensions again form isosceles triangles (which we show below), we transfer a complete set of $(i+1)$-term sequences to λ^{i+1}-length sub-segments of the bases of those triangles, building an I_{i+1}^n that matches our geometric construction. (See Figure 6.)

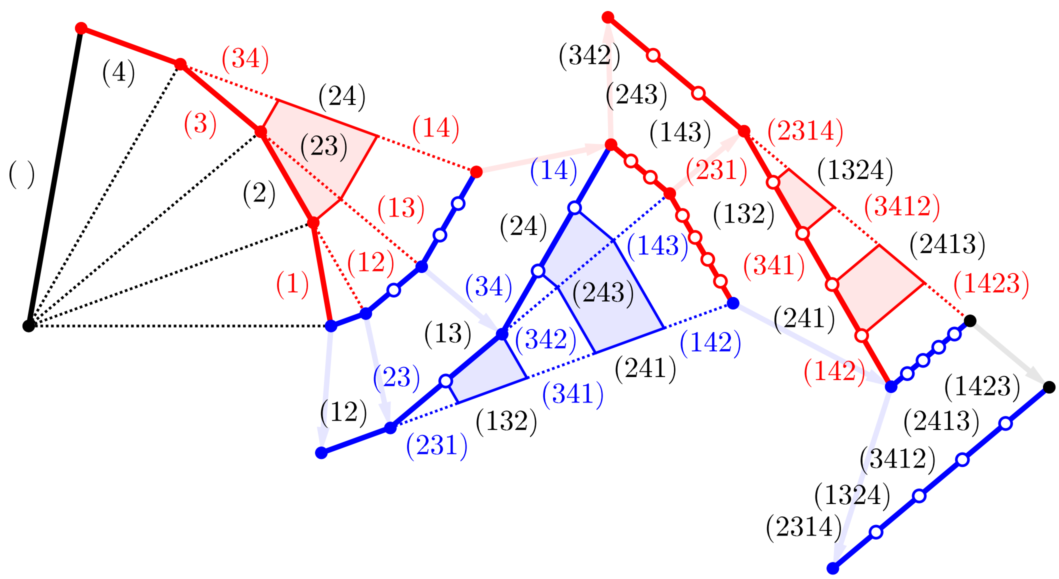


FIGURE 6. Adorning all I_i^4 with i-term up-down sequences of [4].

We see, then, that every i-term up-down sequence of $[n]$ appears exactly once somewhere on I_i^n, so that its length satisfies (10).

As a final detail, we establish the isosceles nature of the triangles by proving that (starting with $j = 0$) *any sequence on the j-th segment of any I_i^n has exactly j successors.*

The property certainly holds for $i = 1$; so let's assume the property holds for some $i \geq 1$. To keep the index arithmetic sane, we structure our argument as rolling refinement of the successor-counting recursion result (8), which we can state thusly:

$$\boxed{k\text{-th successor of an } i\text{-term sequence with } j \text{ successors has } n-i-j+k-1 \text{ successors}}$$

By our induction hypothesis, this becomes:

$$\boxed{k\text{-th successor of any sequence on the } j\text{-th segment of } I_i^n \text{ has } n-i-j+k-1 \text{ successors}}$$

By construction, such a k-th successor adorns an extension of the k-th prior segment of I_i^n; defining $p := j - k$, we have:

$$\boxed{\text{a sequence on any extension of the } p\text{-th segment of } I_i^n \text{ has } n-i-p-1 \text{ successors}}$$

Finally, sequences on extensions of the p-th segment of I_i^n transfer to the $(n-i-1-p)$-th segment[3] of I_{i+1}. Defining $q := n-i-1-p$:

$$\boxed{\text{any sequence on the } q\text{-th segment of } I_{i+1}^n \text{ has } q \text{ successors}}$$

This extends the required successor property to the segments of I_{i+1}^n, completing the induction and our overall proof. □

4. Notes

- We have recaptured André's theorem that $\sec\theta + \tan\theta$ is the exponential generating function for the numbers z_i. See [2] and [4].

- In Chaikovsky's geometric construction of polygonal approximations to the involute pinwheel, binomial coefficients appear in essentially the same way they appeared in our Section 2.1, from considering partial sums of segment lengths. There is a corresponding sequence-based construction to match that of our zig-zag: we need only replace up-down sequences with the up-*up* variety (that is, *monotone increasing* sequences). See Figure 7.

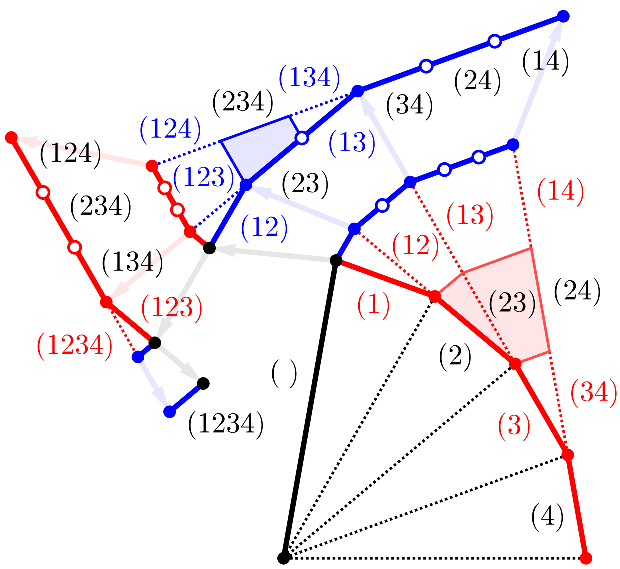


FIGURE 7. Adorning a polygonal pinwheel with up-up sequences of [4].

Like the zig-zag case (but with *far* less effort), one can show that all (and only) i-term up-up sequences of n elements occur on I_i^n. And, like the zig-zag case, one can establish an $\binom{n}{i}$-to-1 correspondence between i-term up-up sequences of n elements

[3]Arithmetic sanity check: The last segment —i.e., the $(n-i)$-th— of I_i^n receives no extensions, but extensions of the next-to-last segment —the $(n-i-1)$-th— have sequences that transfer to the first —the 0-th— segment of I_{i+1}^n. Extensions of *earlier* segments of I_i^n transfer to *later* segments of I_{i+1}.

and i-term up-up *permutations*. (This represents a minor —but satisfying— improvement over Chaikovsky's formulation, as the binomial coefficient appears, not as the result of some inductive calculation, but in its most combinatorially-meaningful capacity.) *Un*like in the zig-zag case, we have uniqueness: there is exactly one i-term up-up permutation, namely, "$(1, 2, ..., i)$"; we have no need, therefore, for an explicit "z_i"-like factor when counting the n-term sequences. Apart from this, the remaining computations of the lengths of the (polygonal or true) involutes are completely analogous across both the pinwheel and zig-zag cases.

- The author gratefully acknowledges the assistance of Robin Chapman of the University of Exeter.

- Now, what about $\csc\theta$ and $\cot\theta$?

References

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